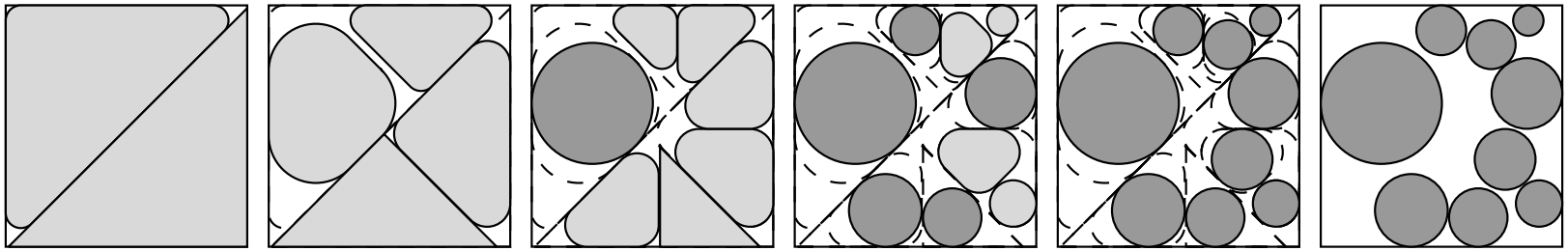
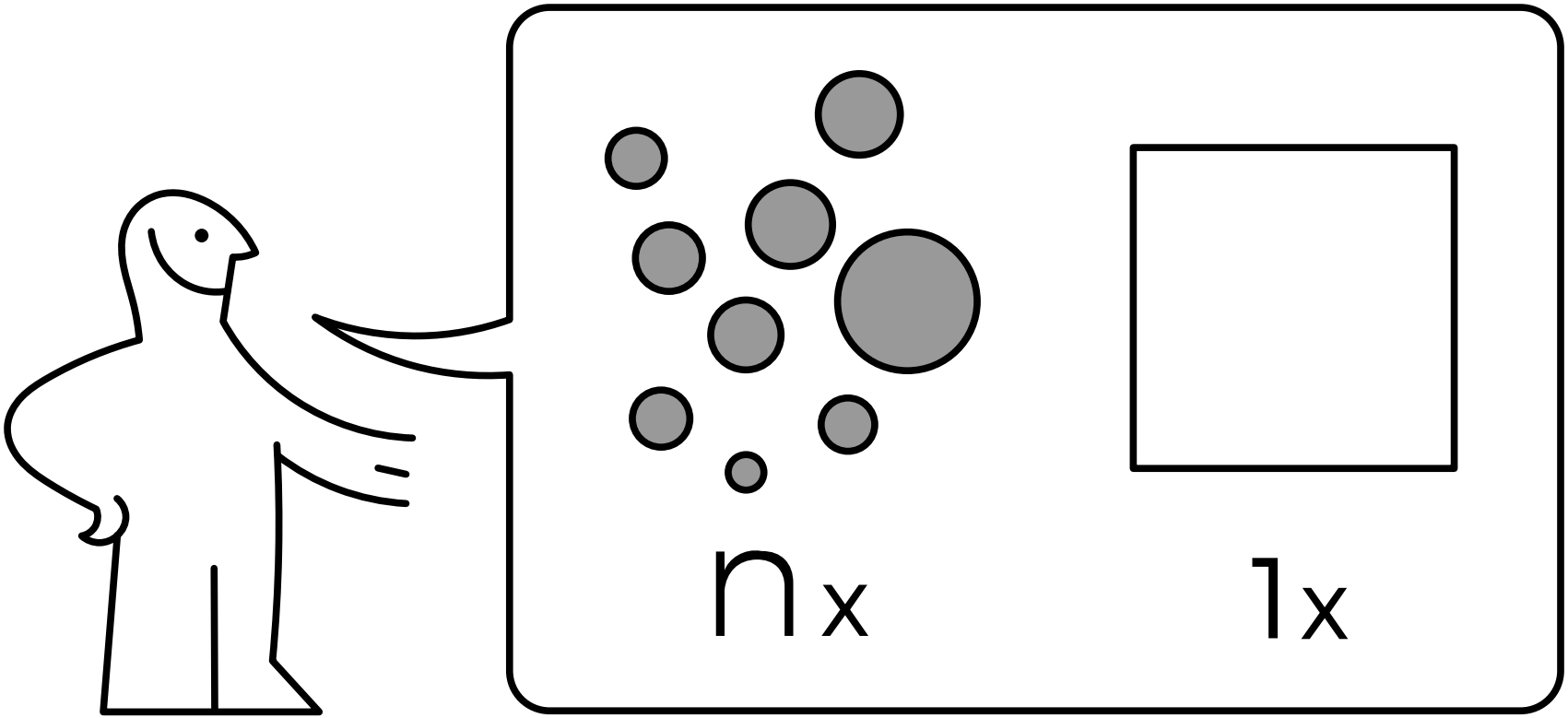


Split Packing: An Algorithm for Packing Circles with up to Critical Density

Sebastian Morr

2016-06-09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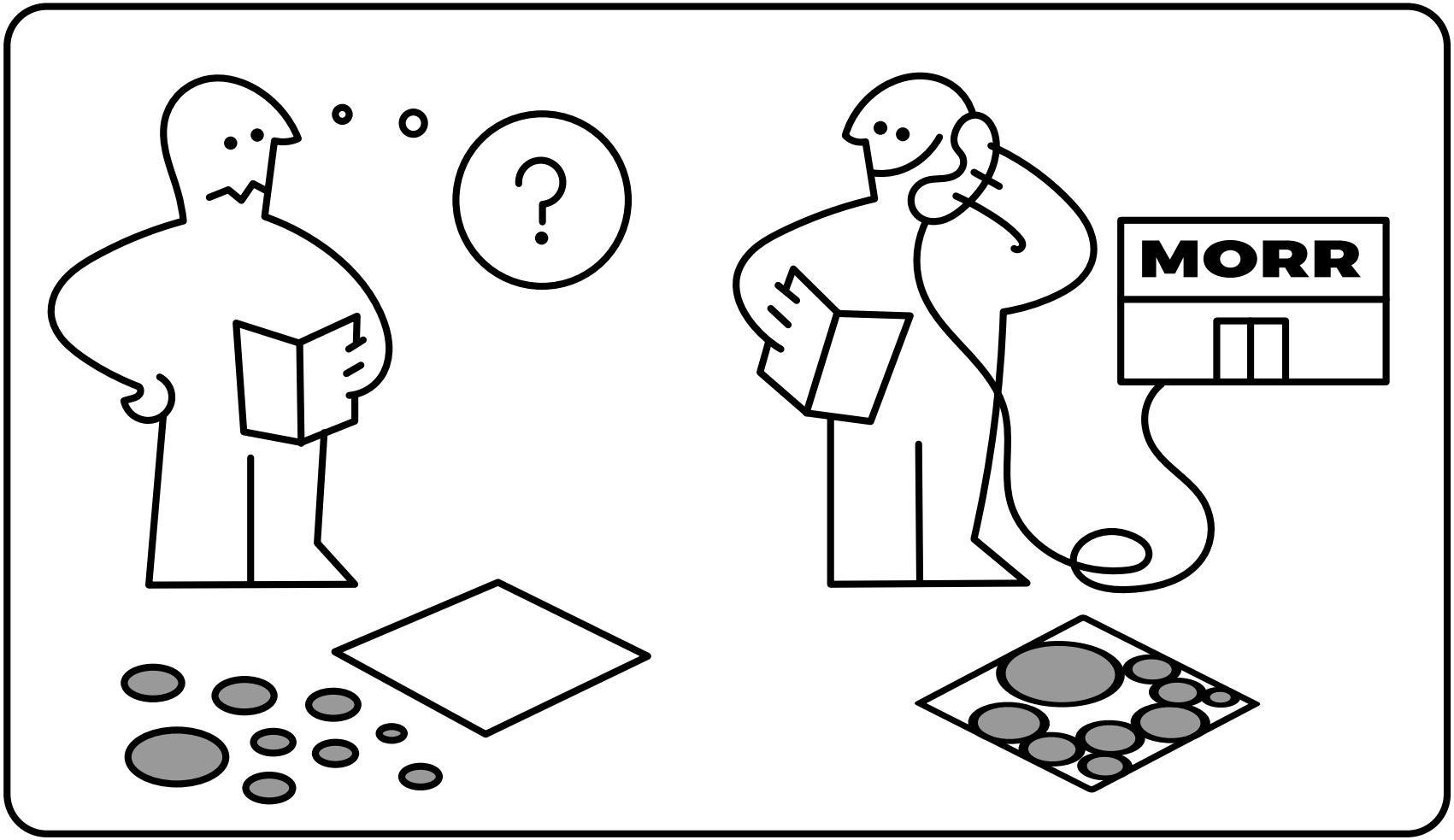
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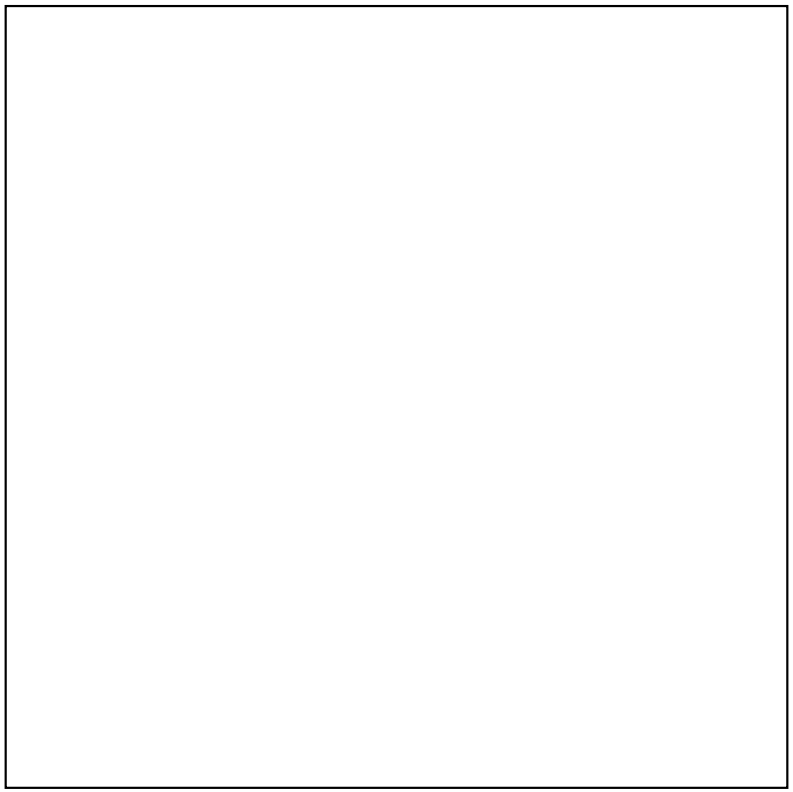
Packing squares in a square

Packing squares in a square



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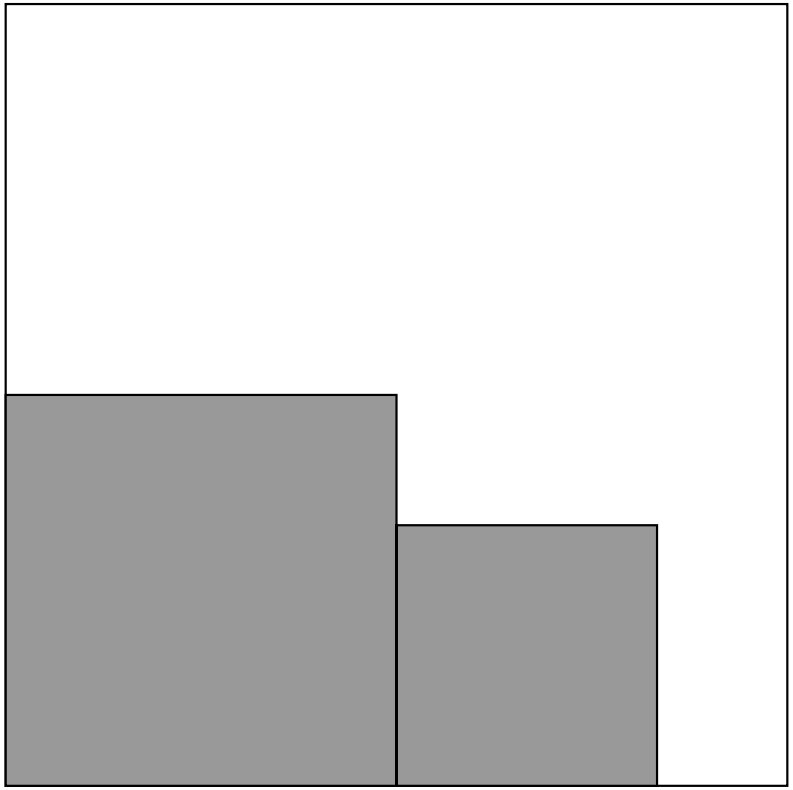

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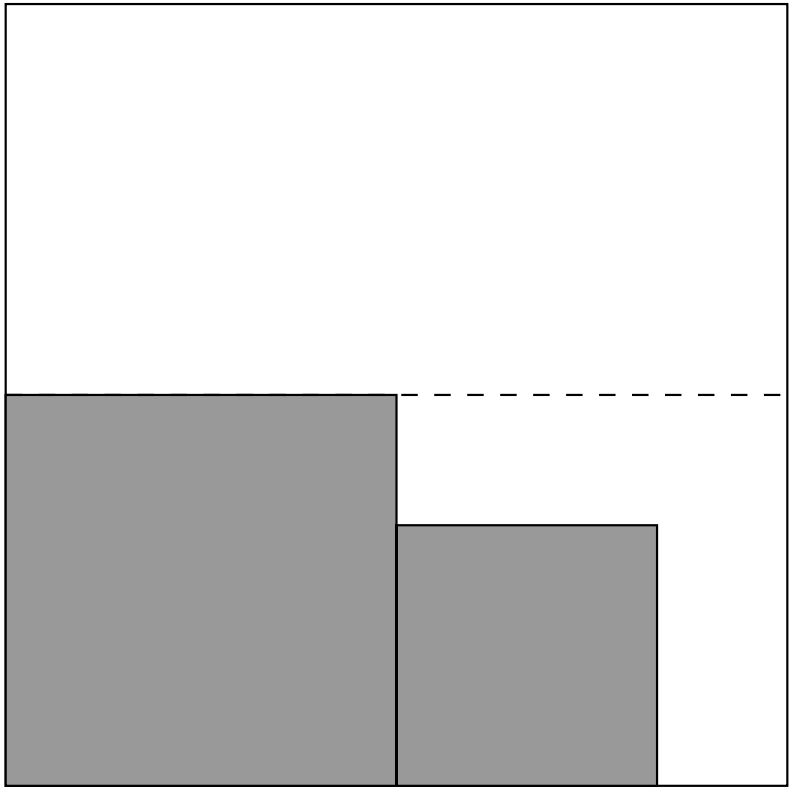

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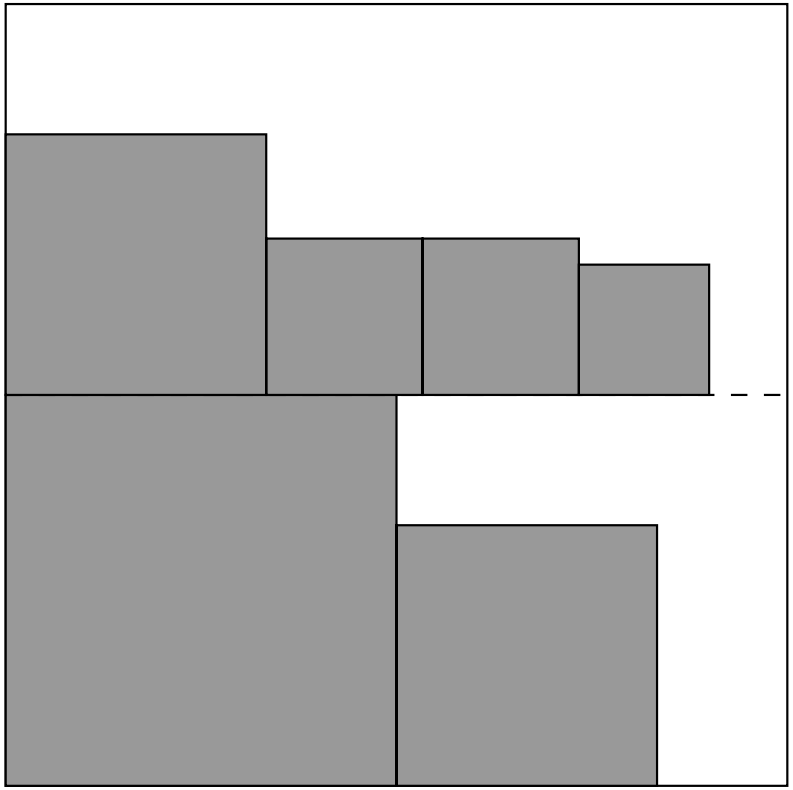

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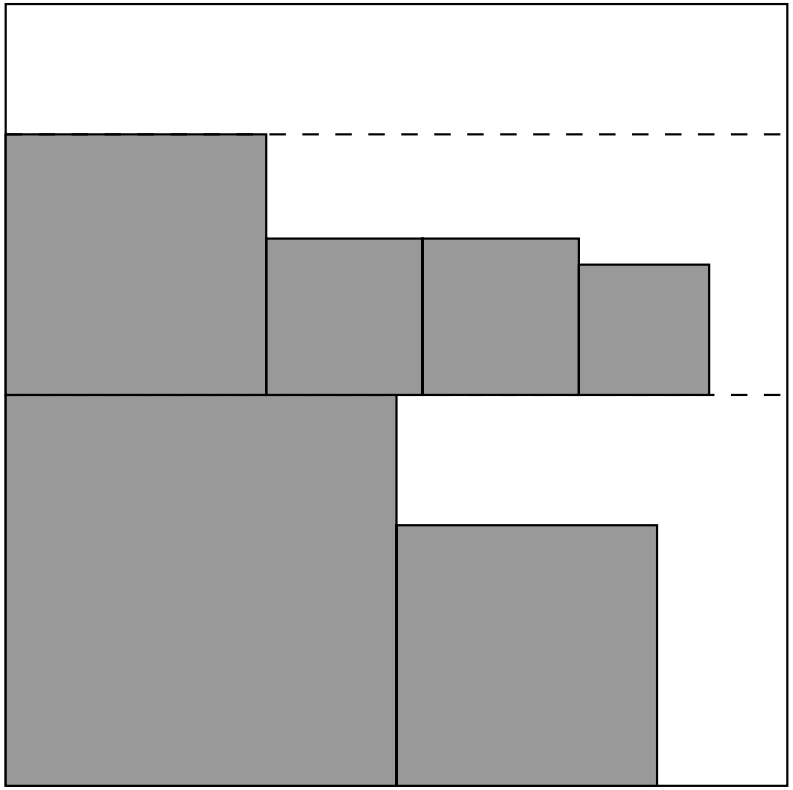

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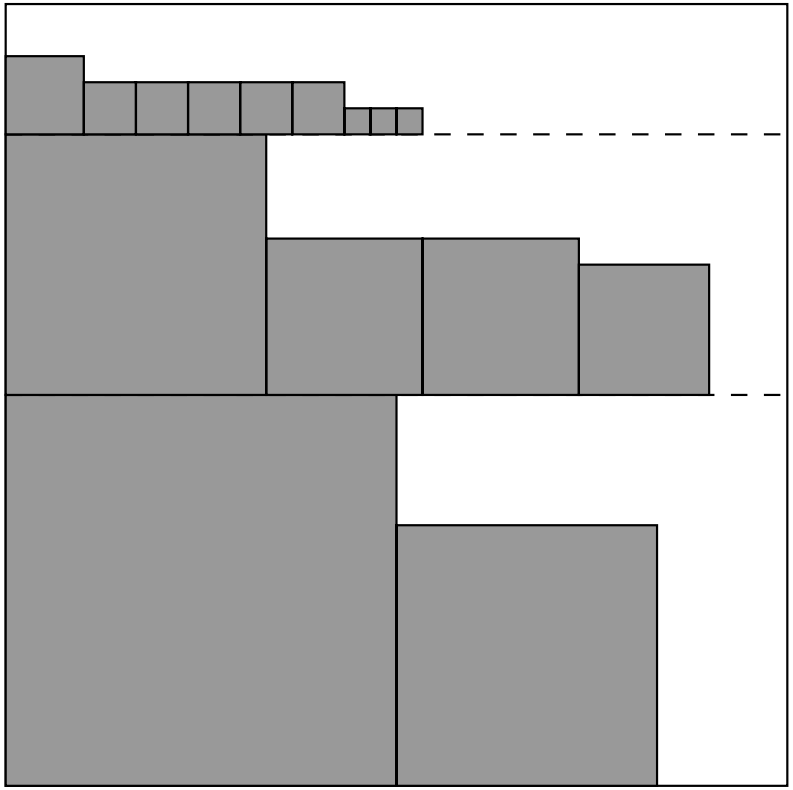

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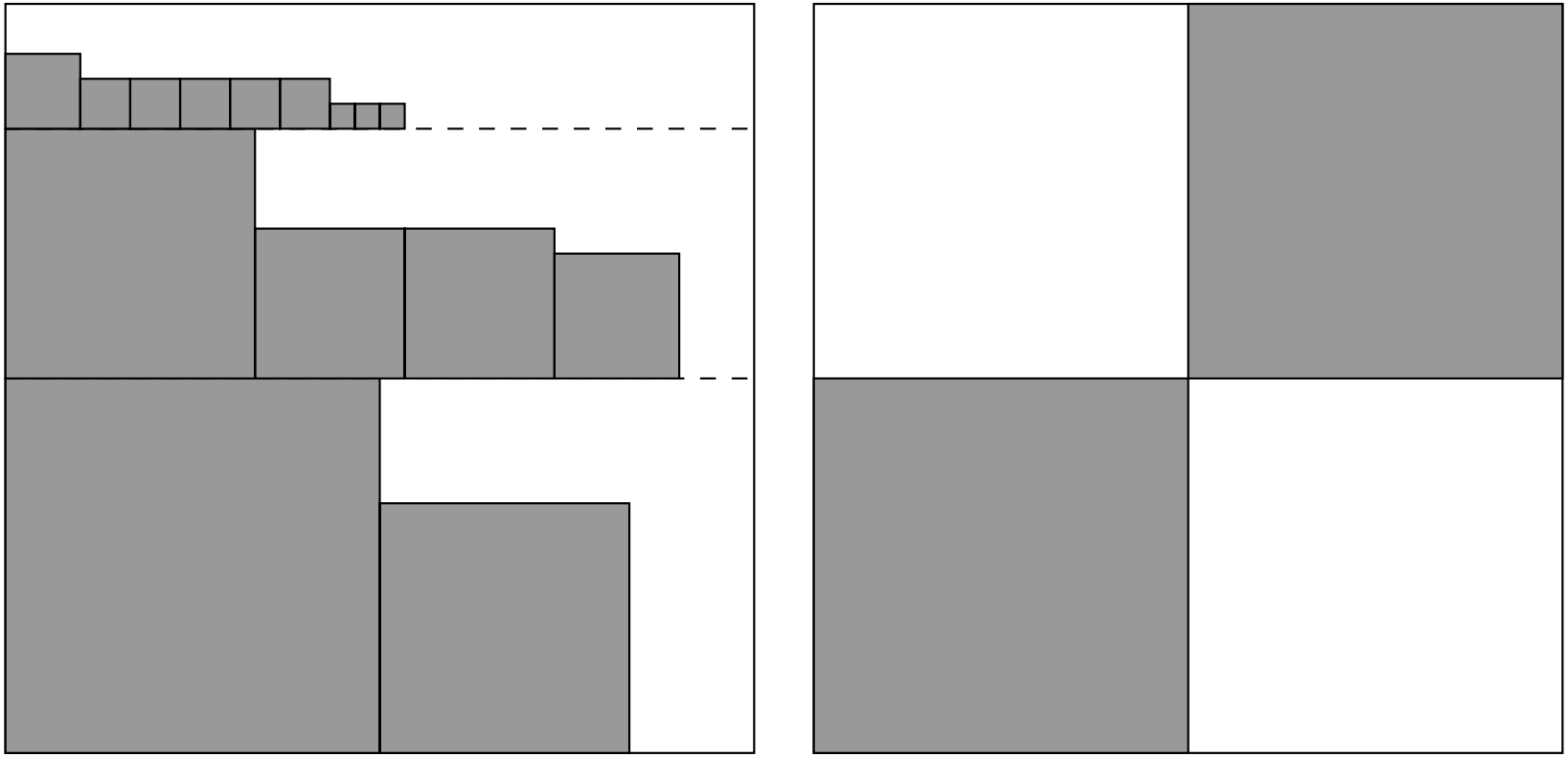

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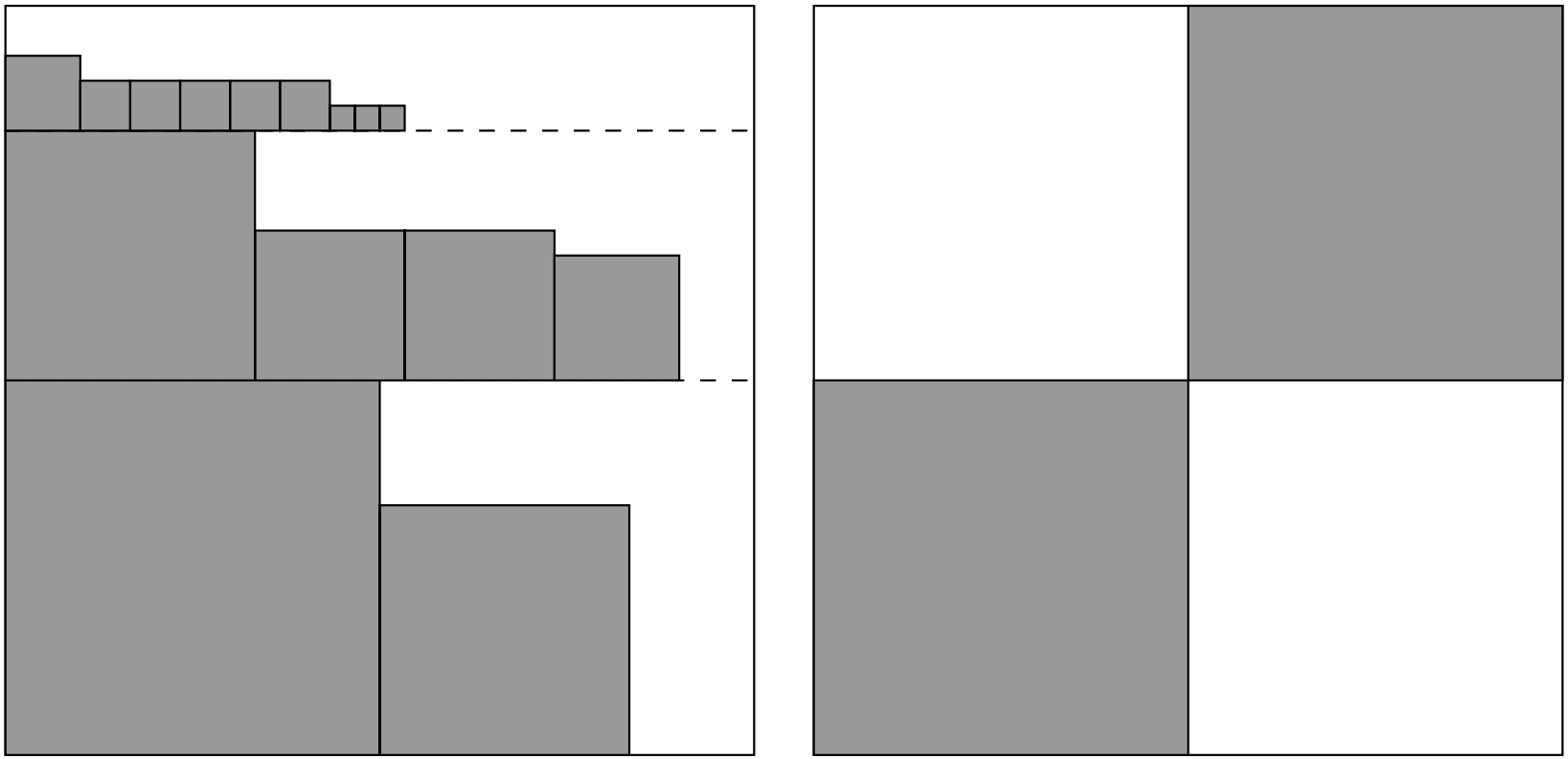


The critical density for packing squares is $1/2$ [Moon & Moser, 1967]

Outline

What about circles?

Critical density for packing circles into a square

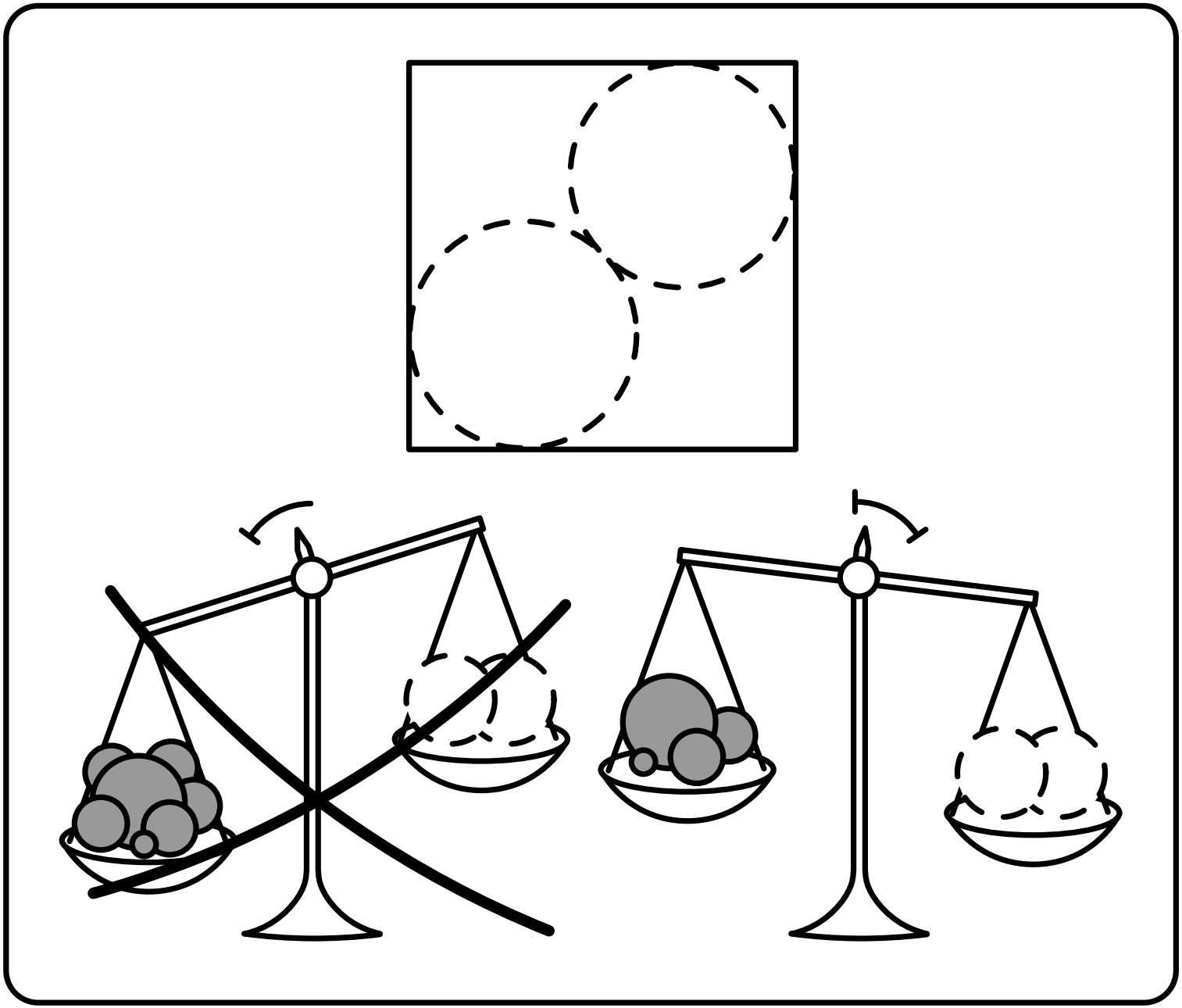
What is the largest a so that any set of circles with a combined area of a can be packed into the unit square?

What about circles?

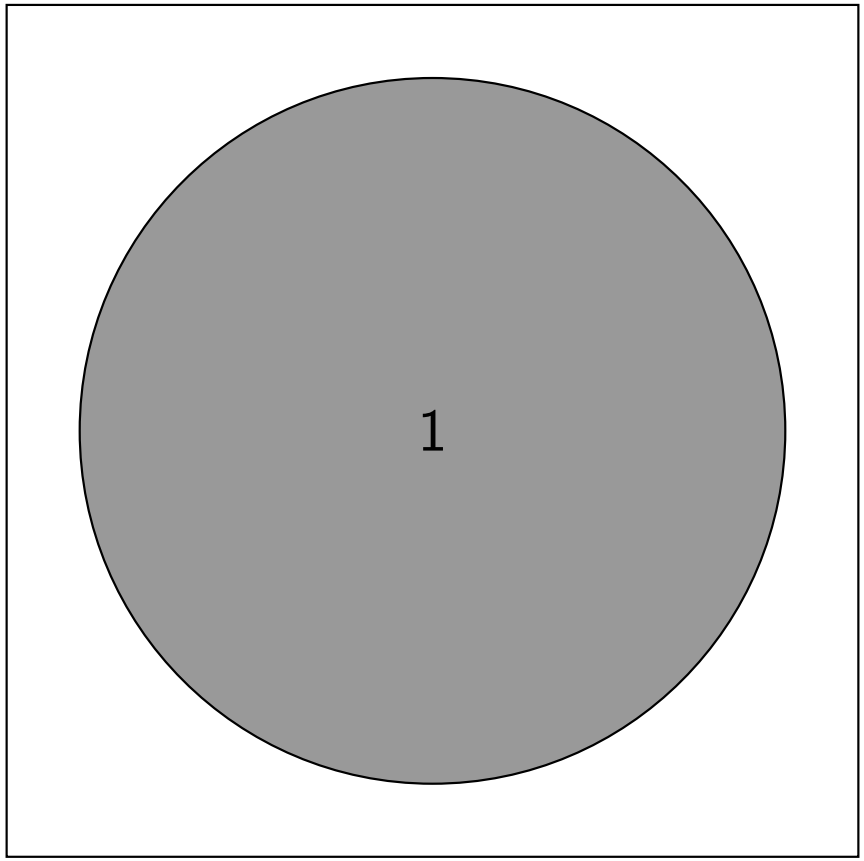
Critical density for packing circles into a square

What is the largest a so that any set of circles with a combined area of a can be packed into the unit square?

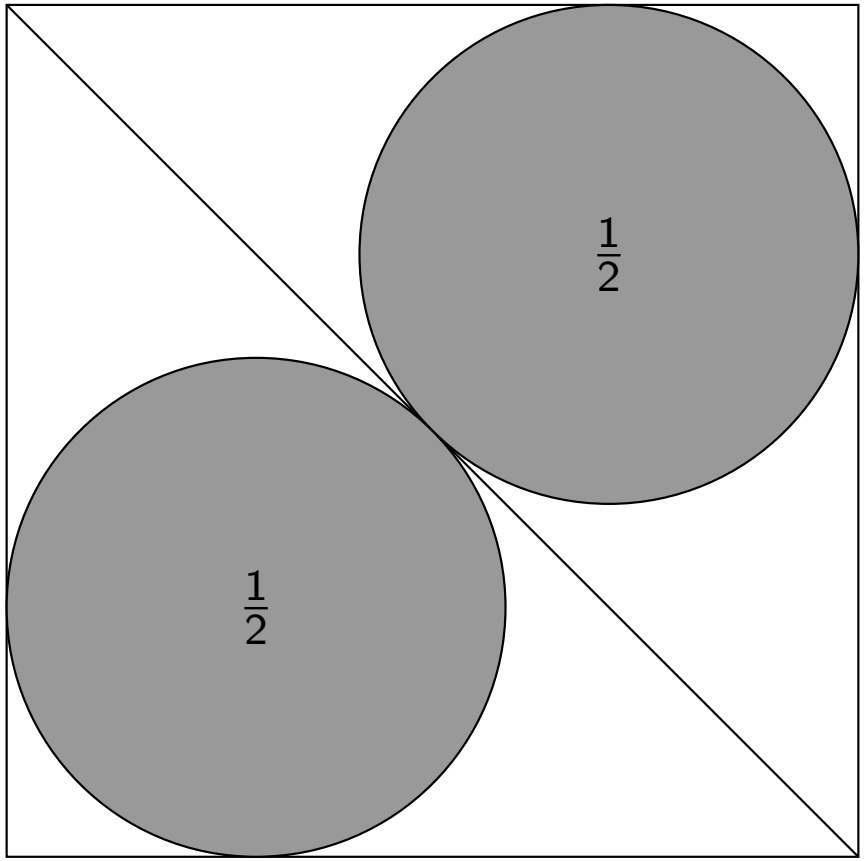
$\rightarrow$ Now: Constructive proof!



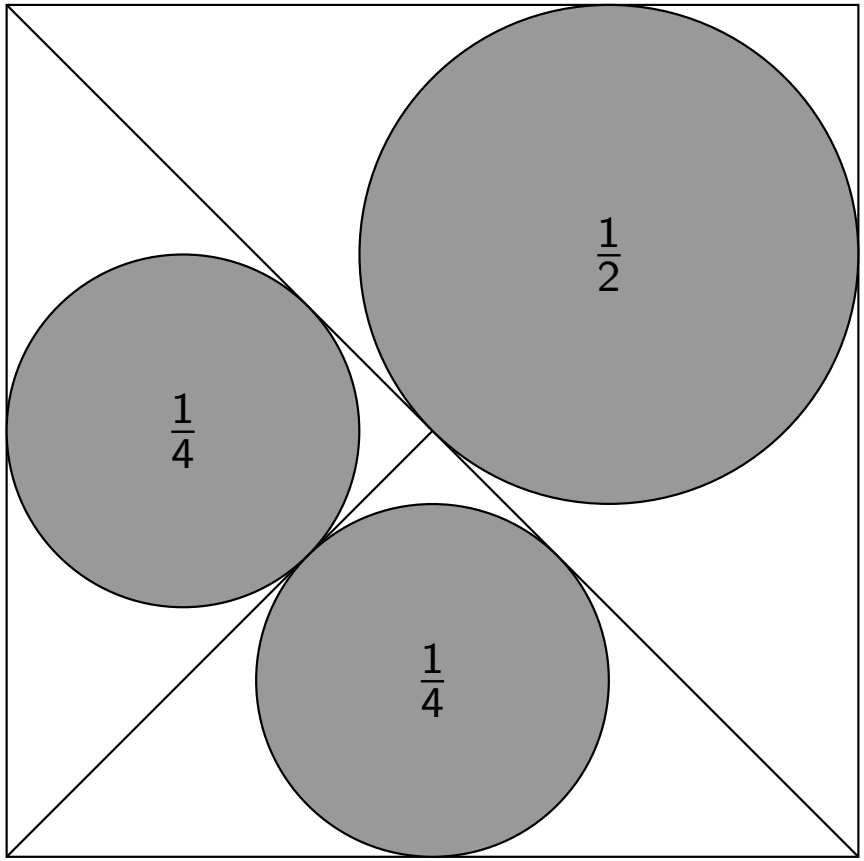
Observation: Splitting in half is easy



Observation: Splitting in half is easy



Observation: Splitting in half is easy



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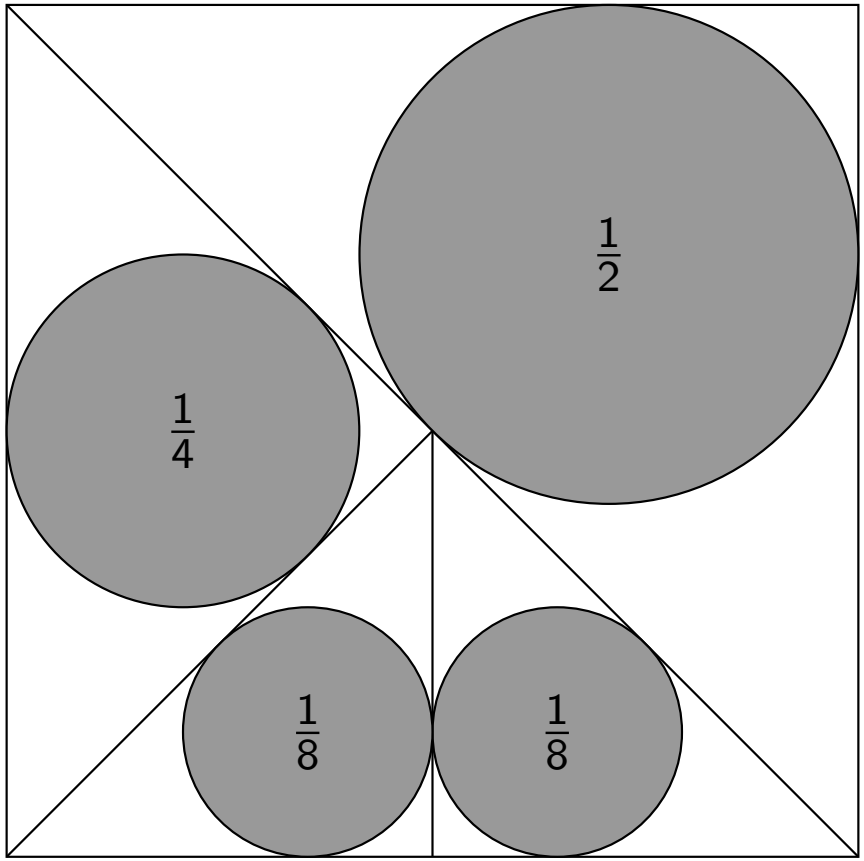

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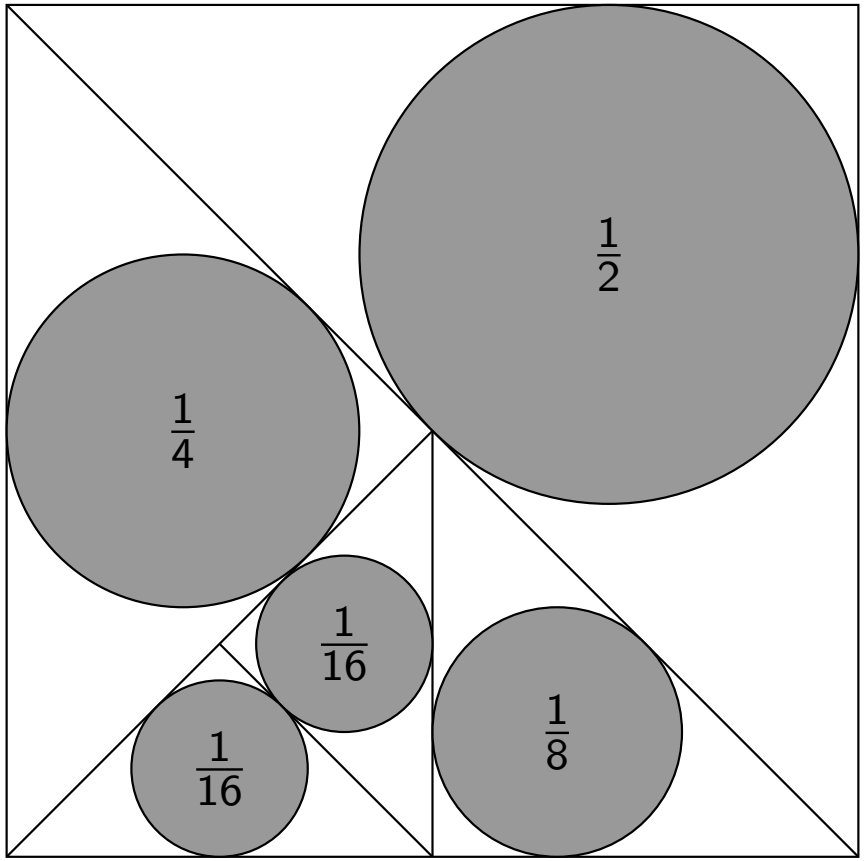

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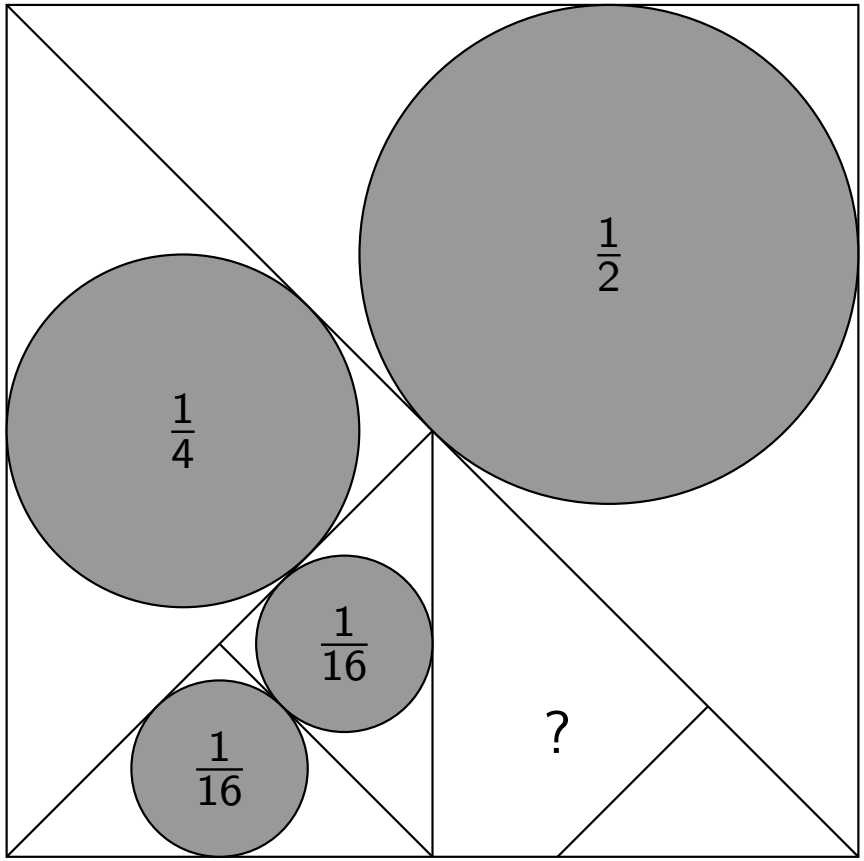

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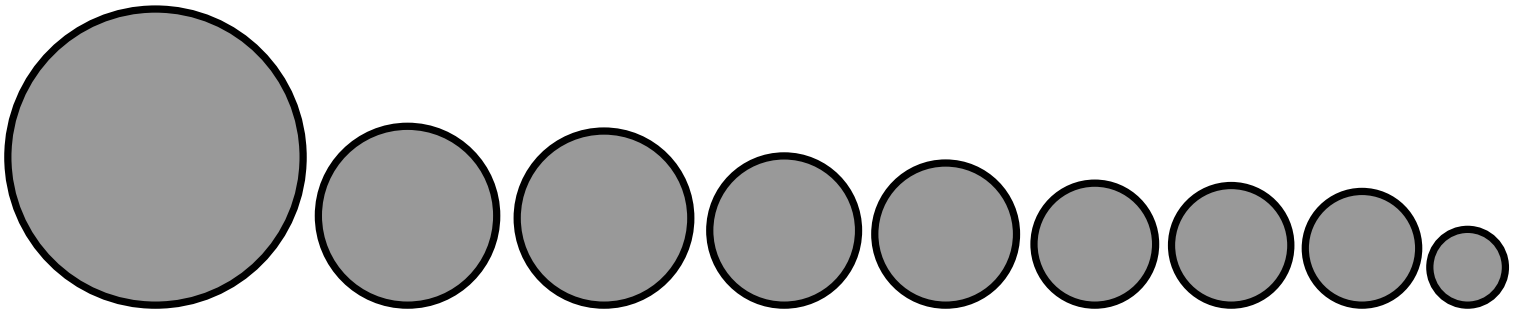

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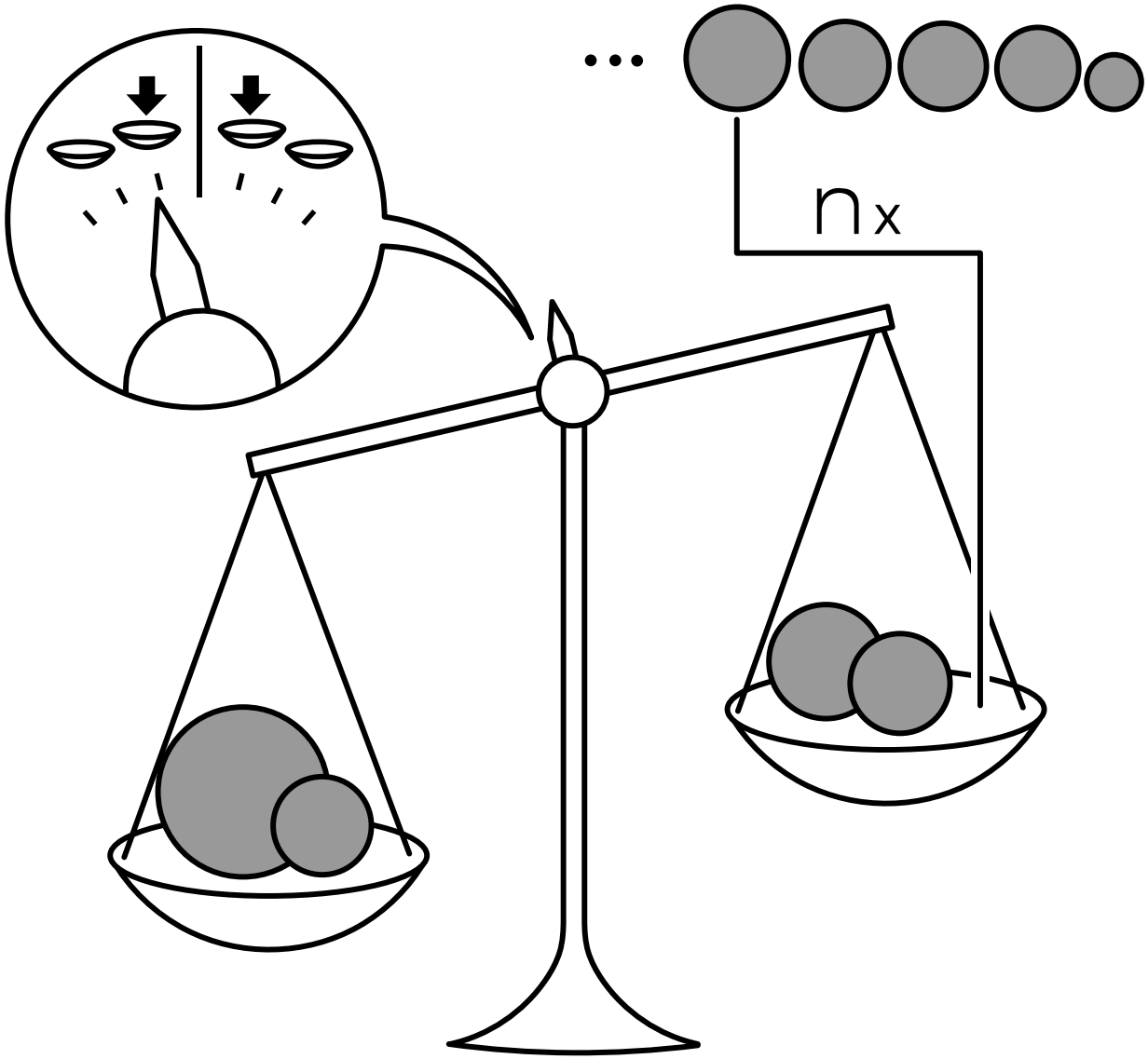
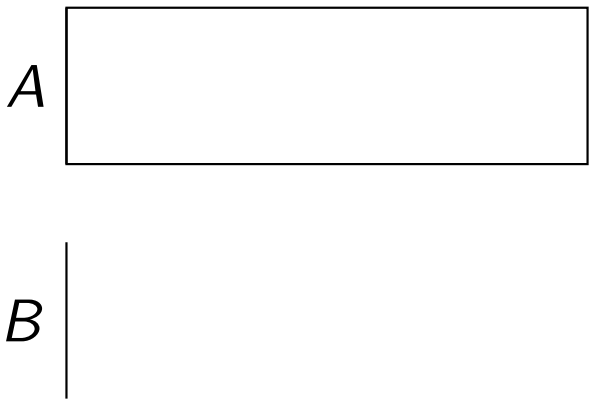


Greedy splitting

Greedy splitting



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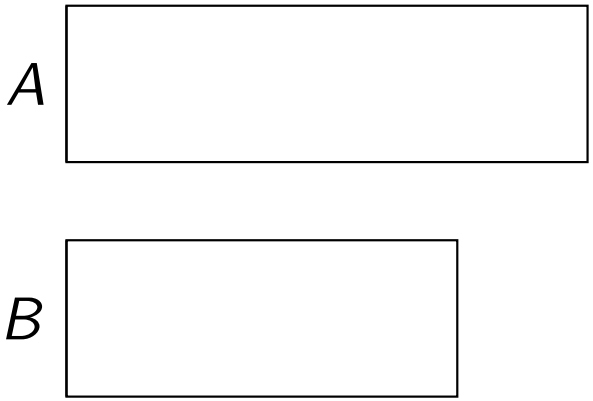

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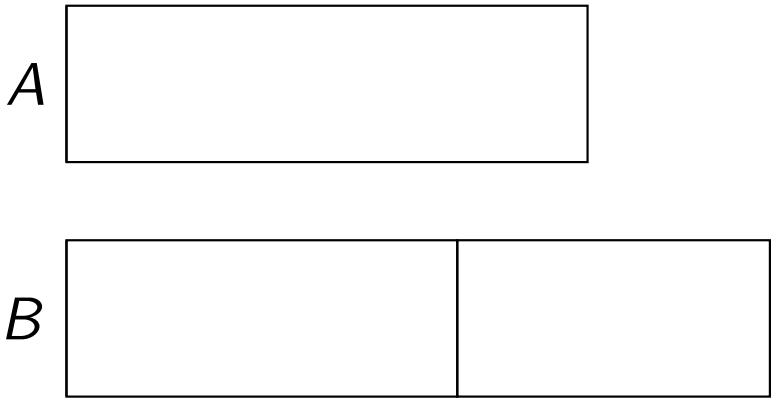

Greedy splitting

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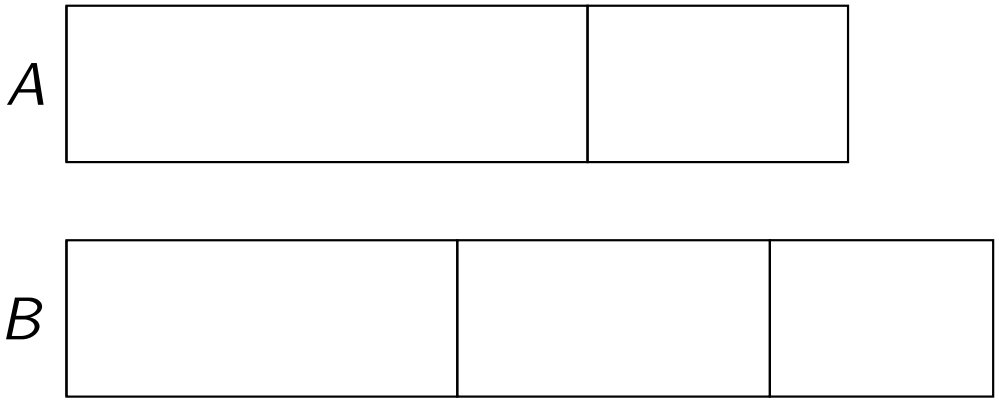
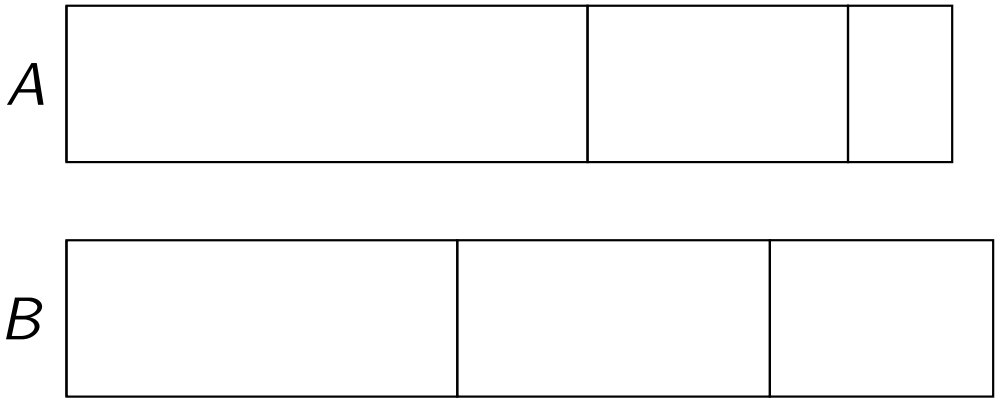
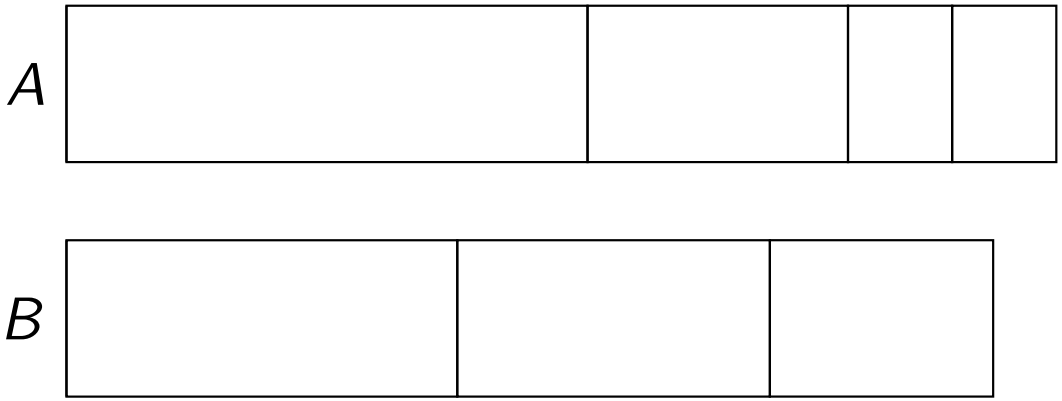


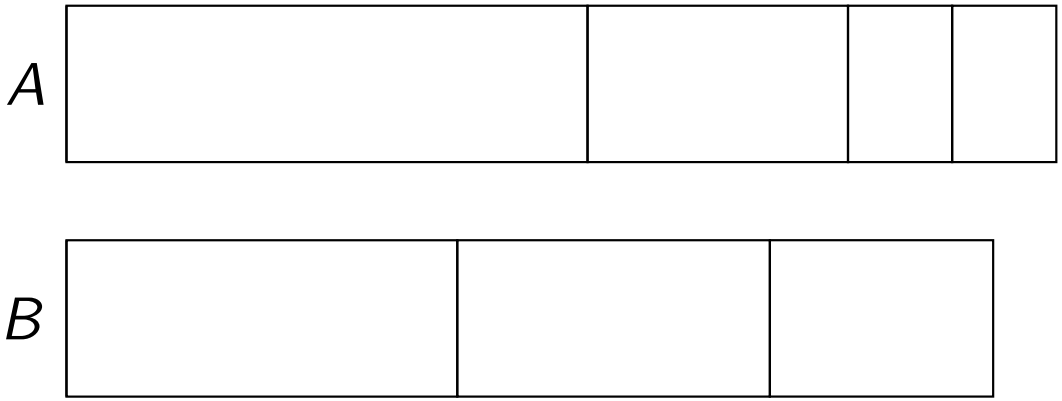
Greedy splitting



Greedy splitting



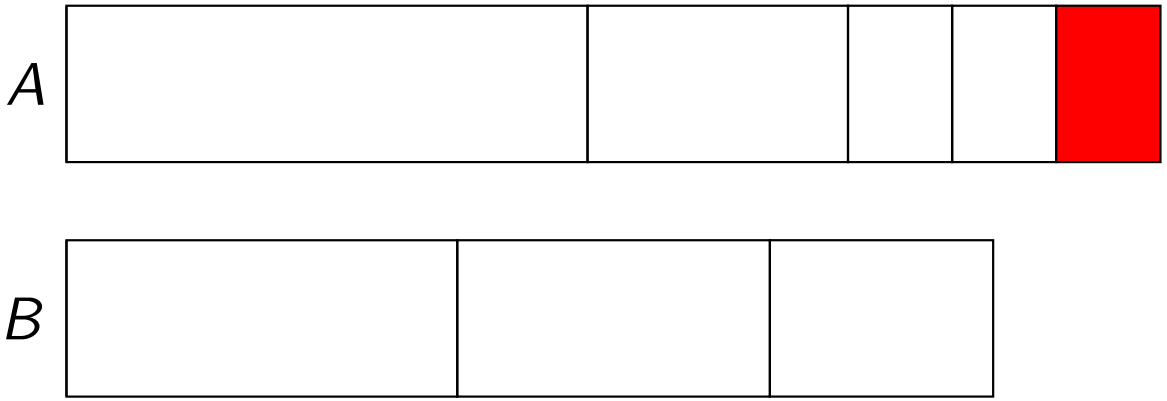
Greedy splitting



Split property:

All elements of larger group $\geq$ groups' difference.

Greedy splitting



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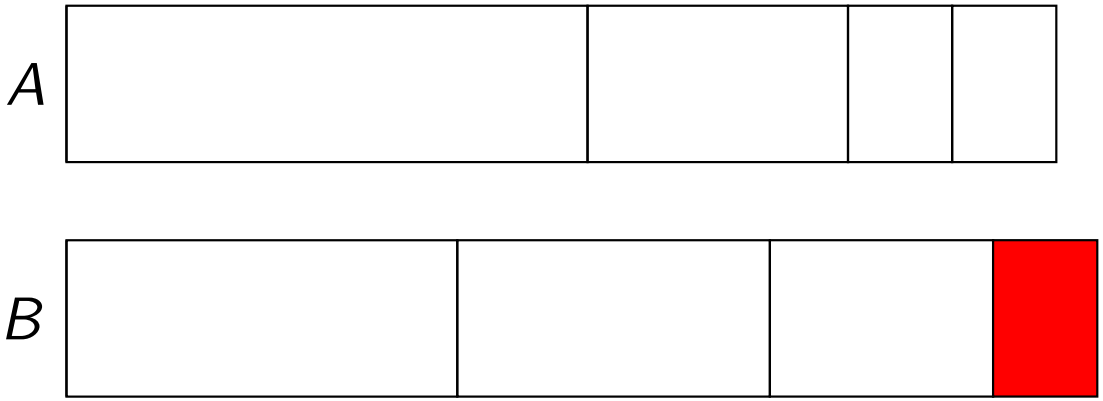

Split property:

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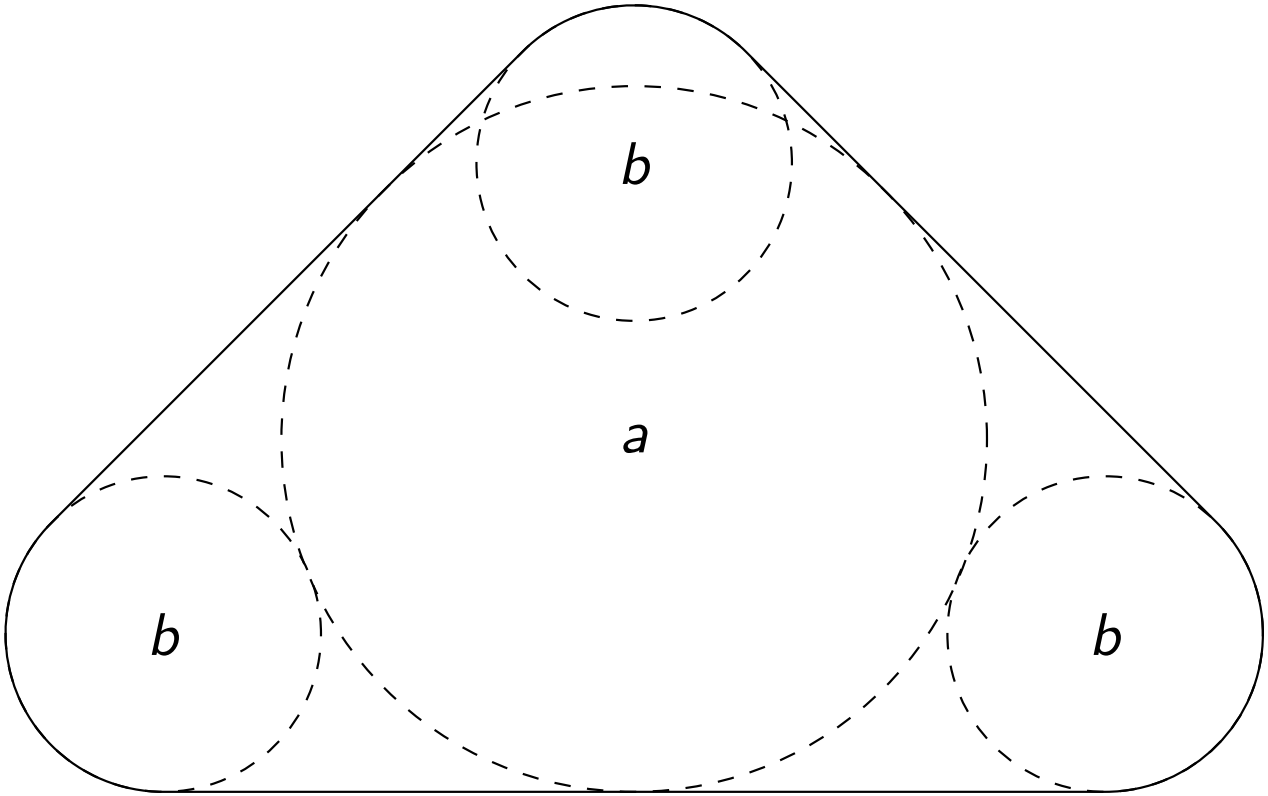
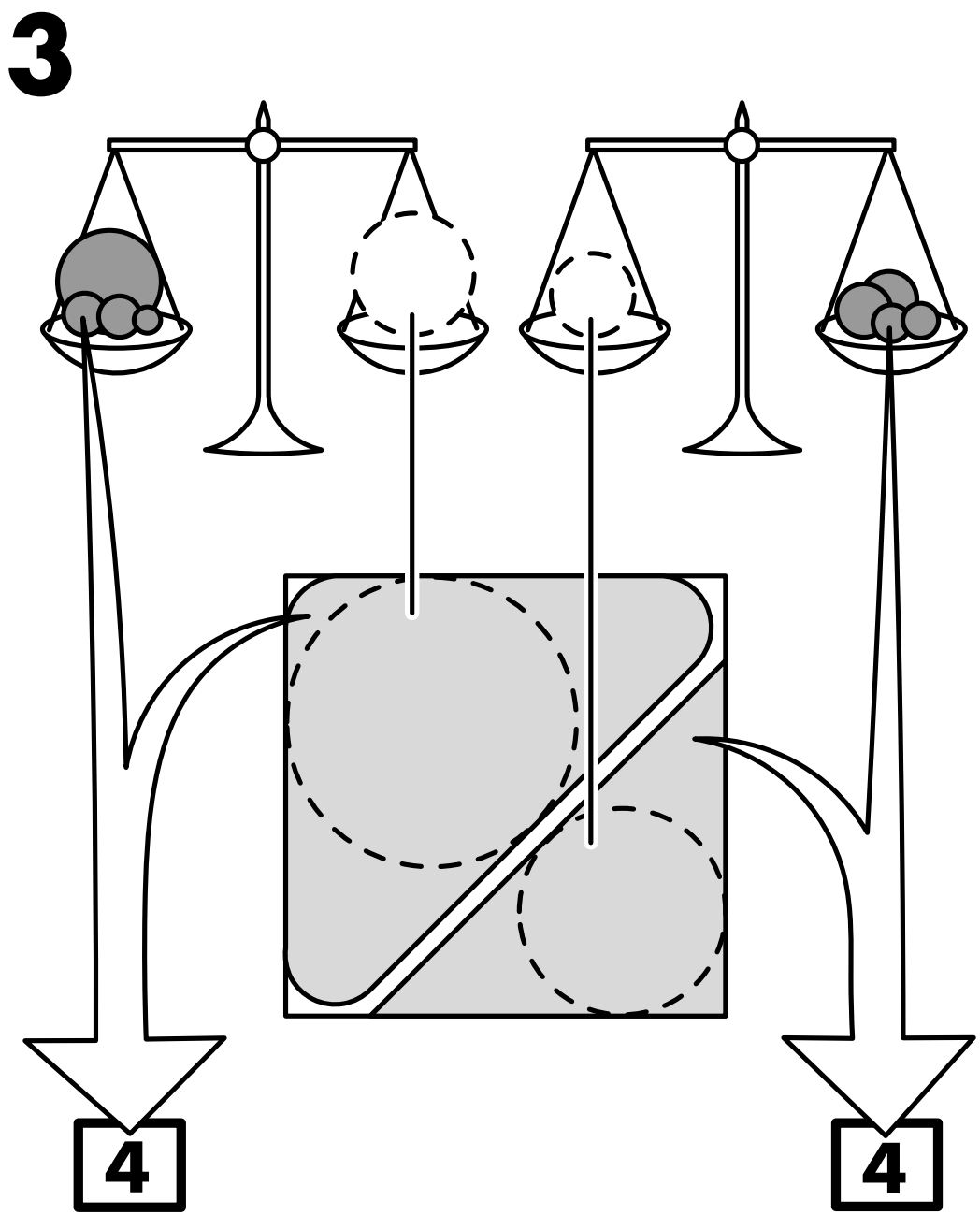
An (a, b)-hat

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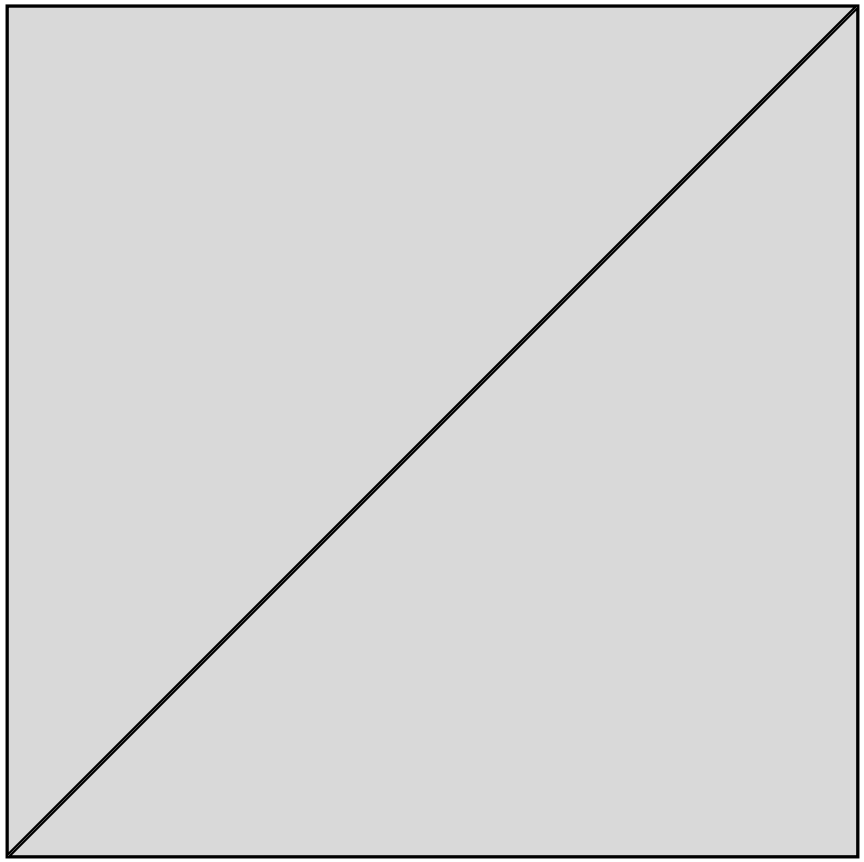

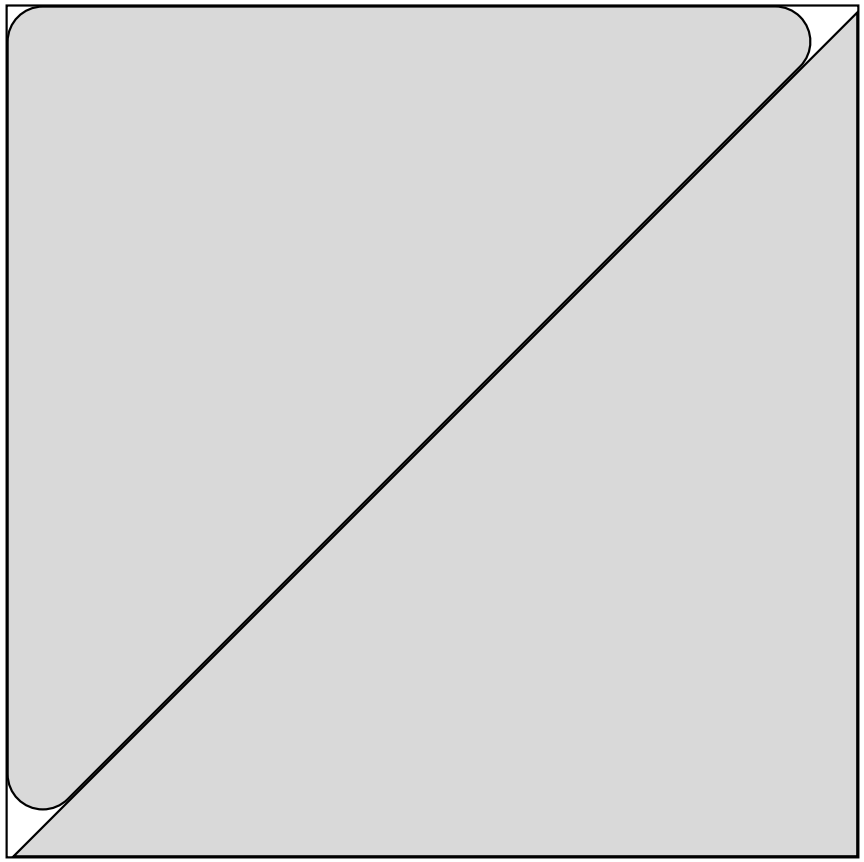

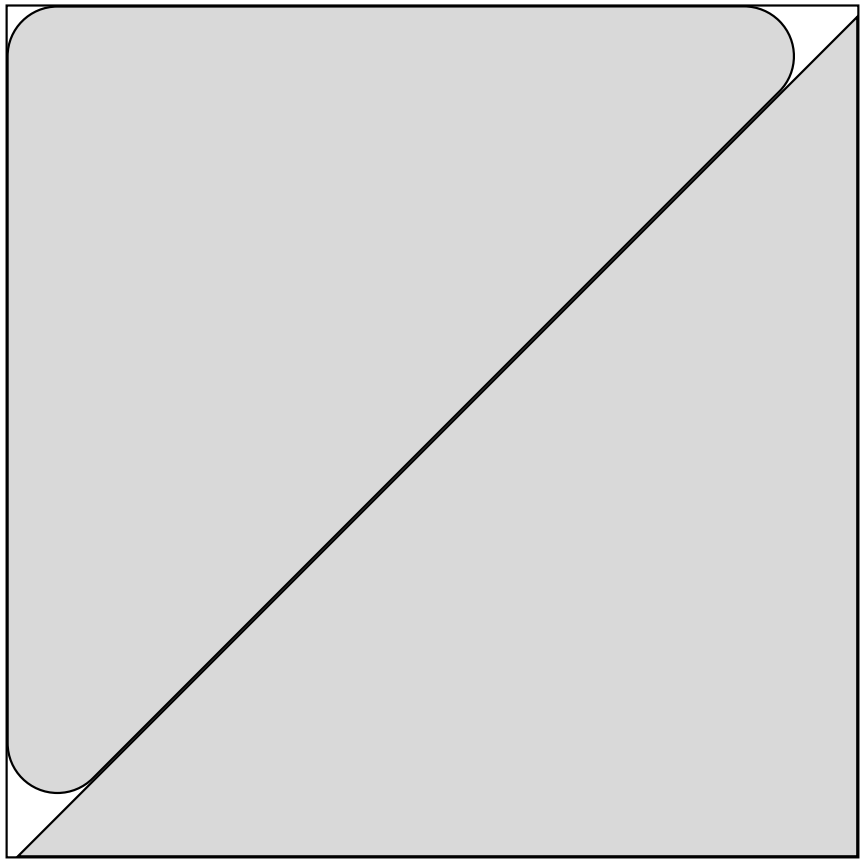

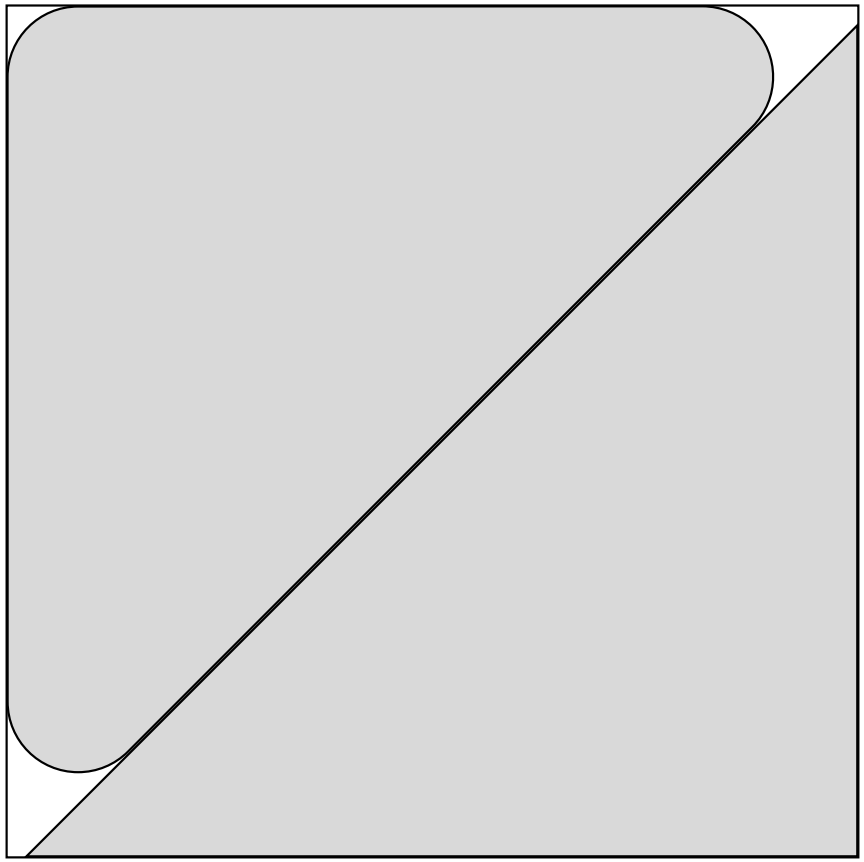

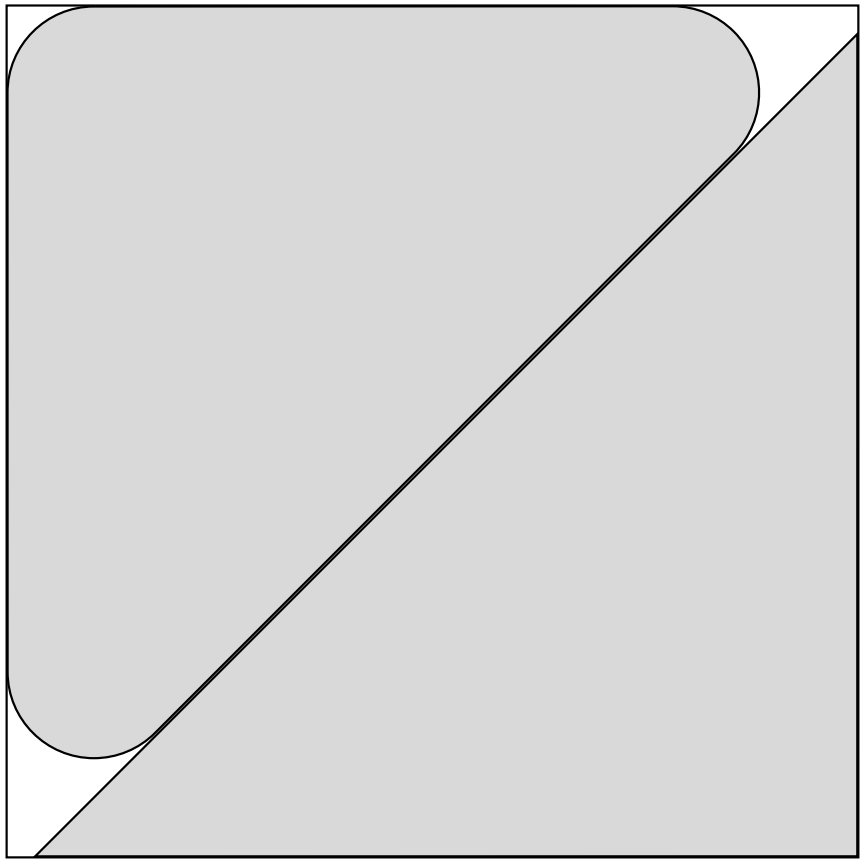

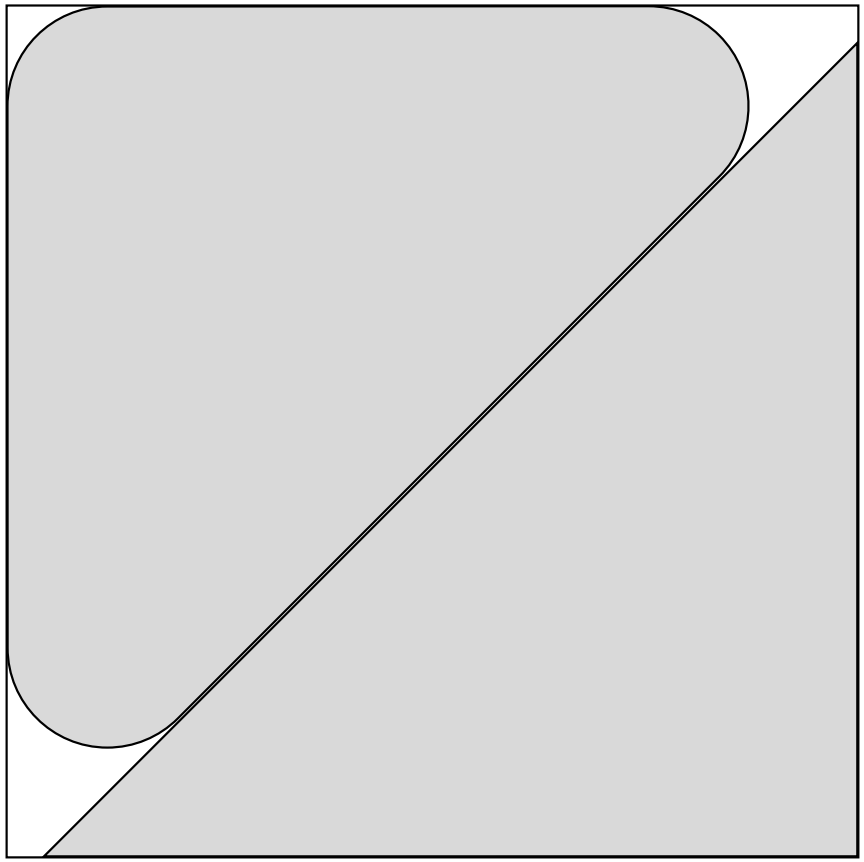

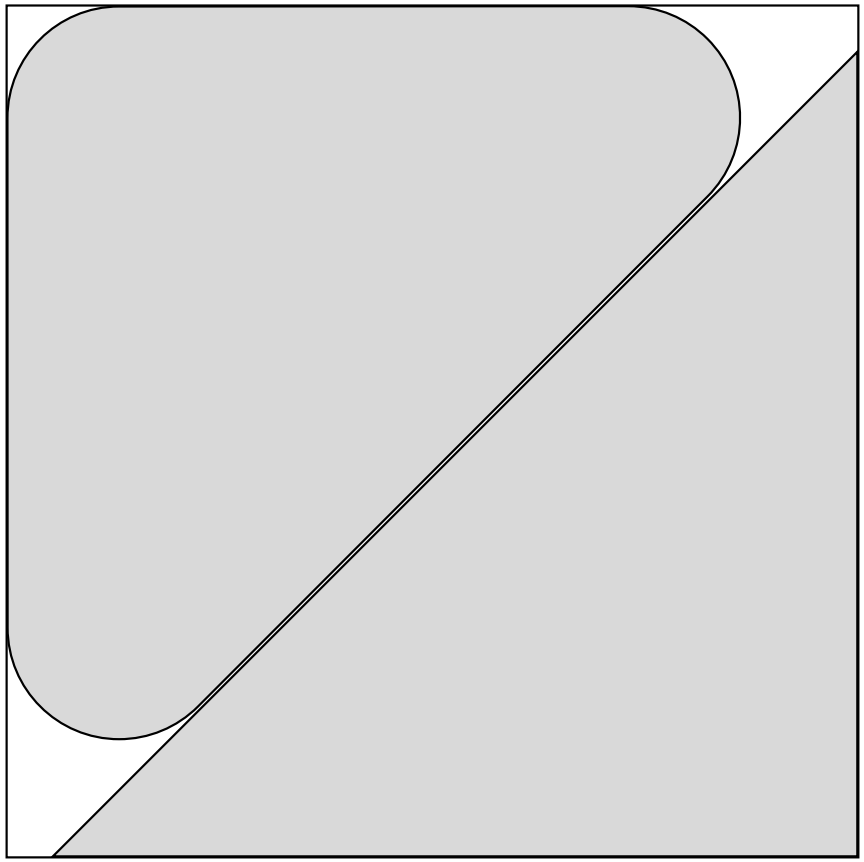

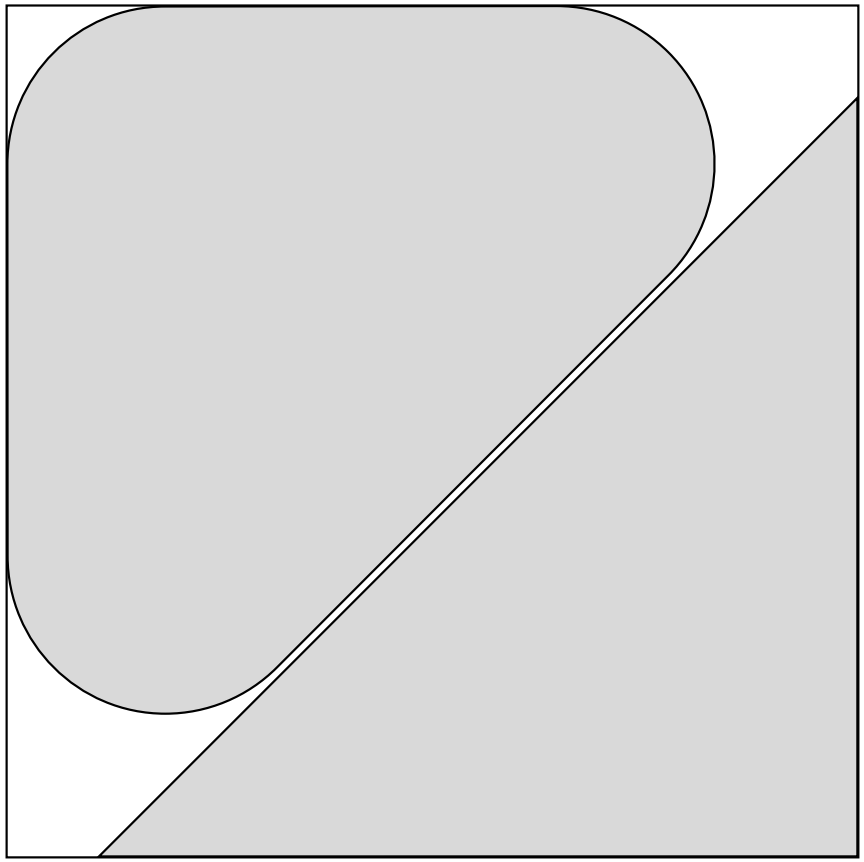

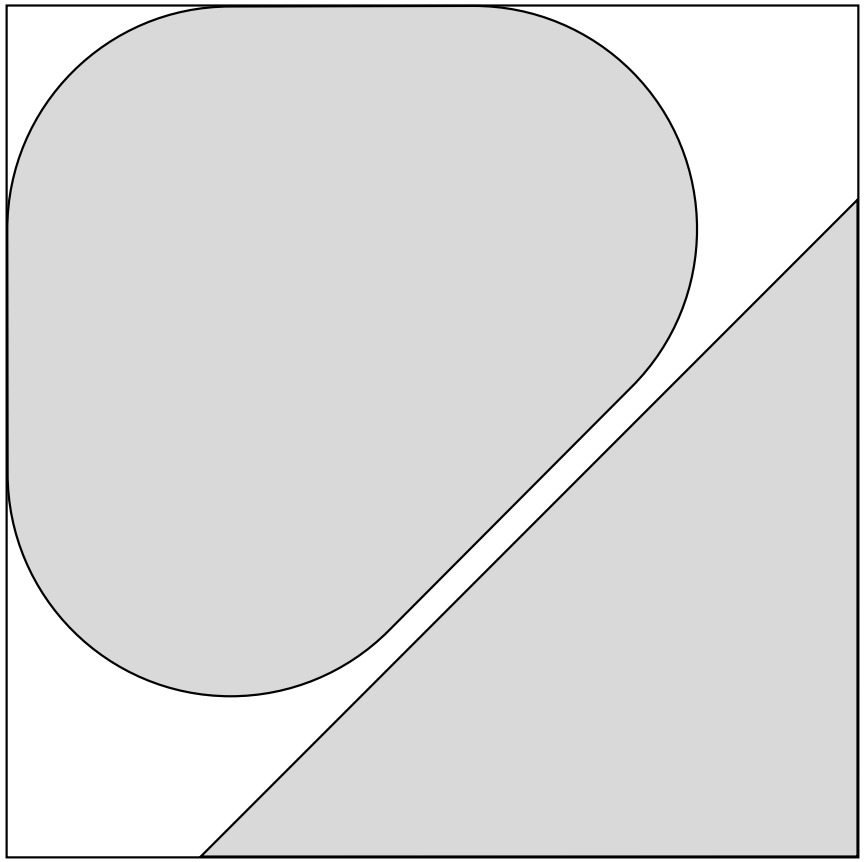

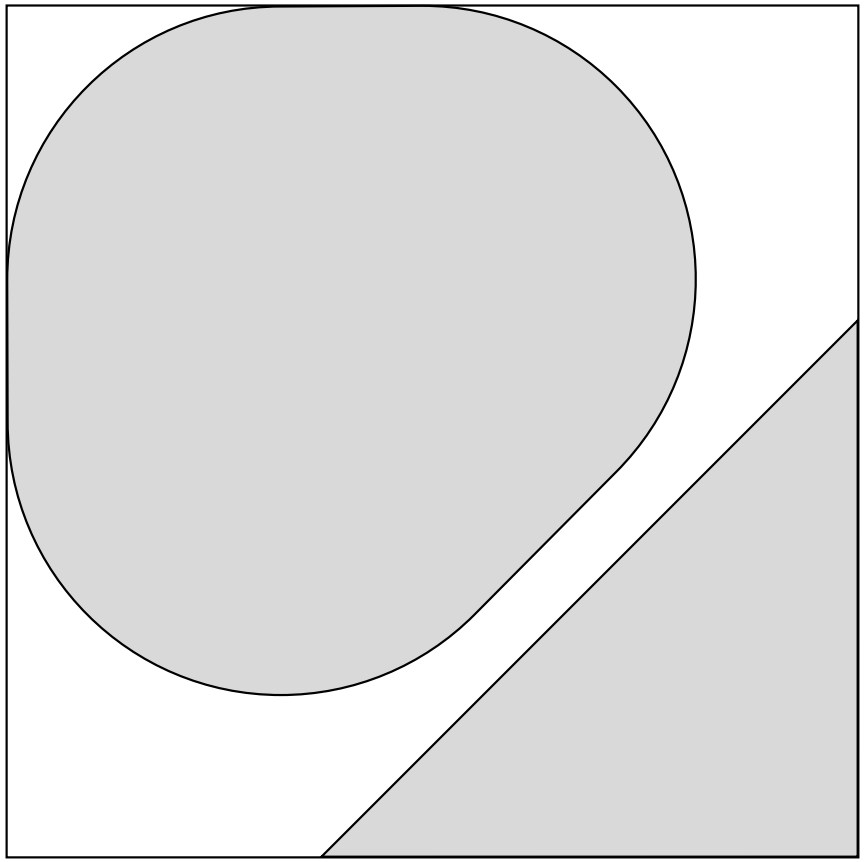

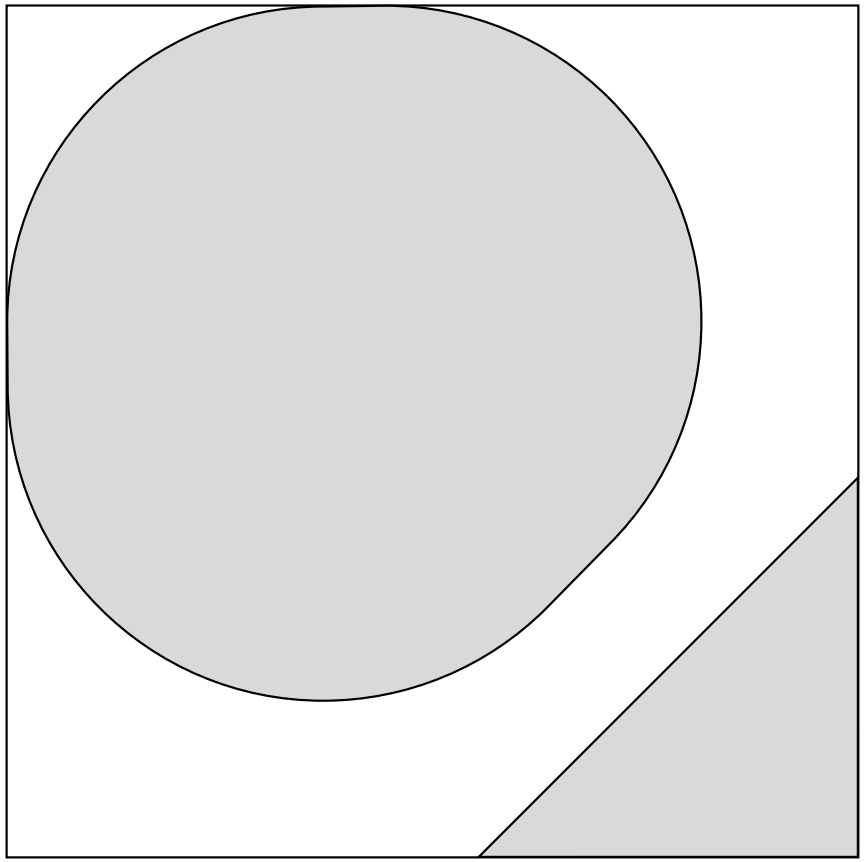

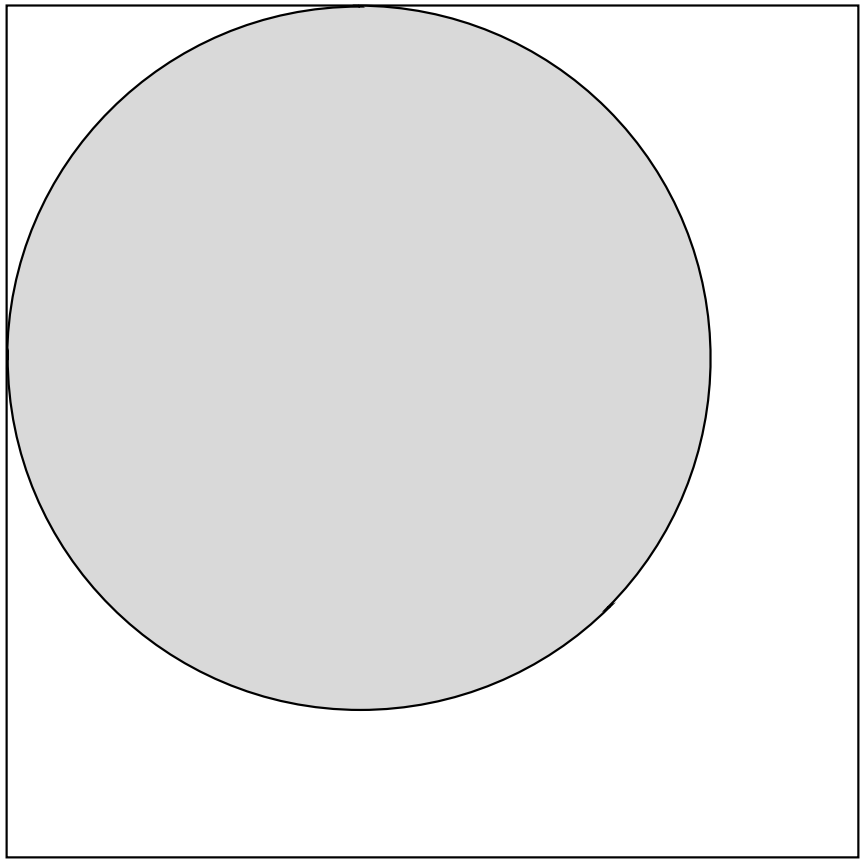

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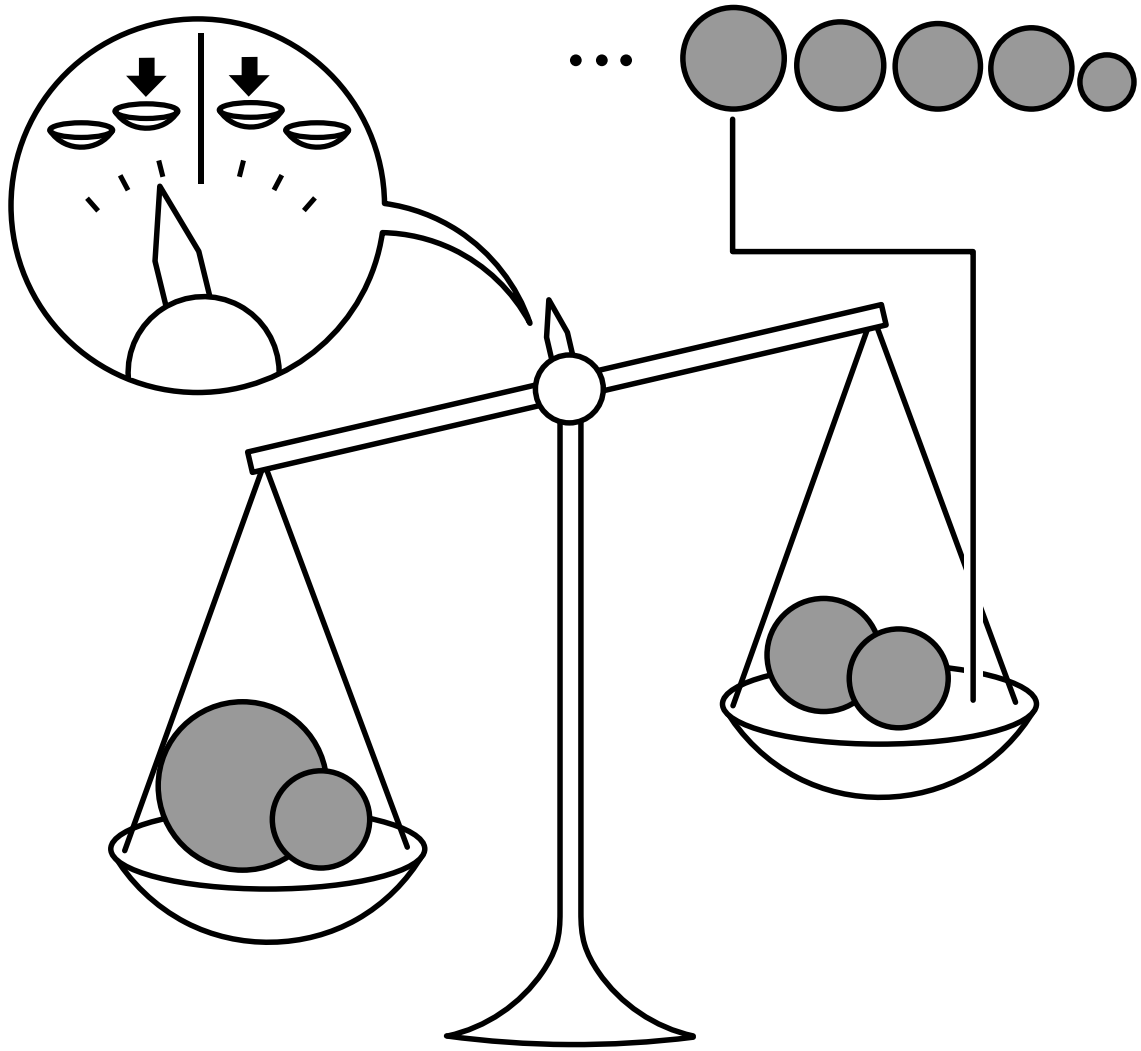

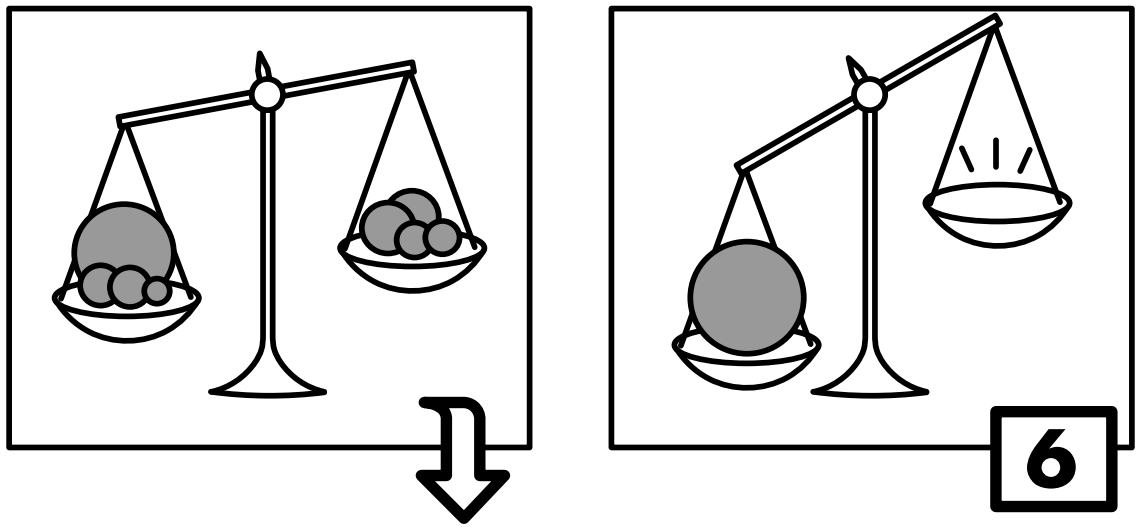
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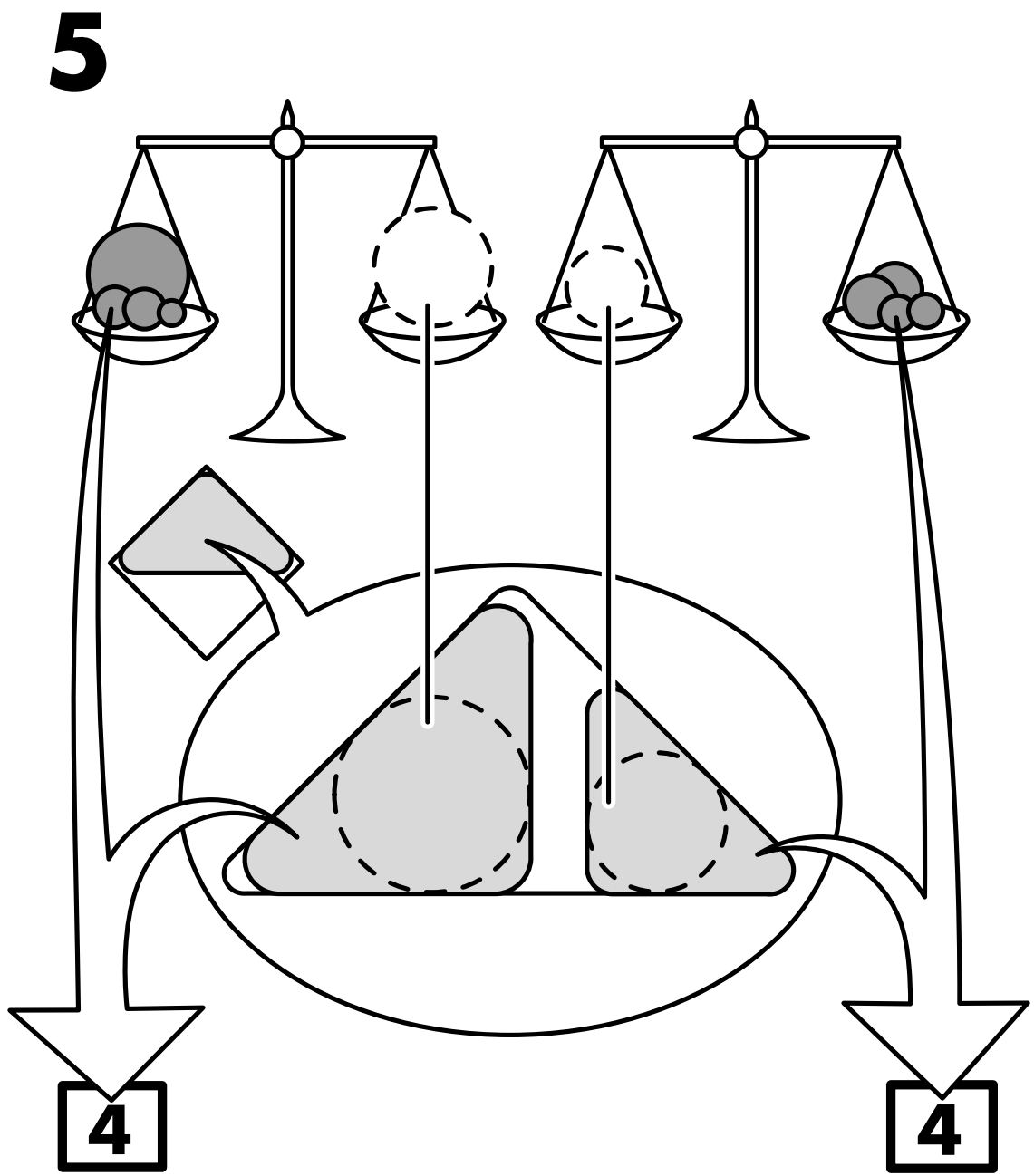
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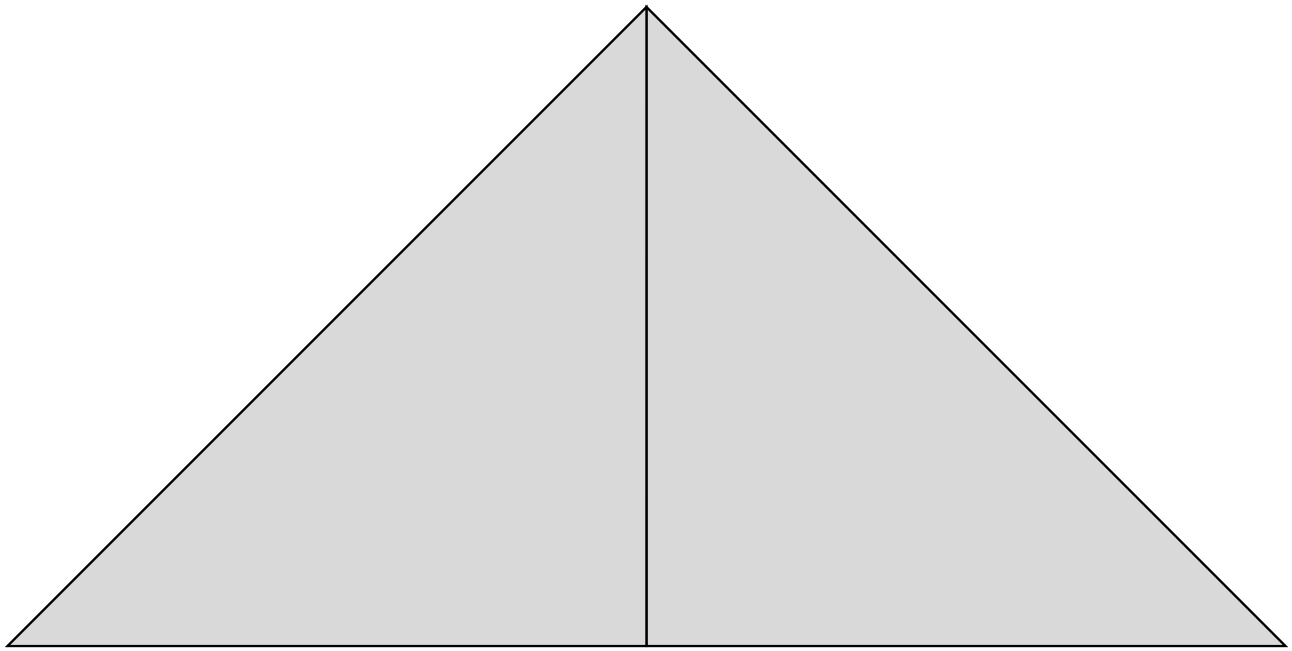
Packing hats in a hat



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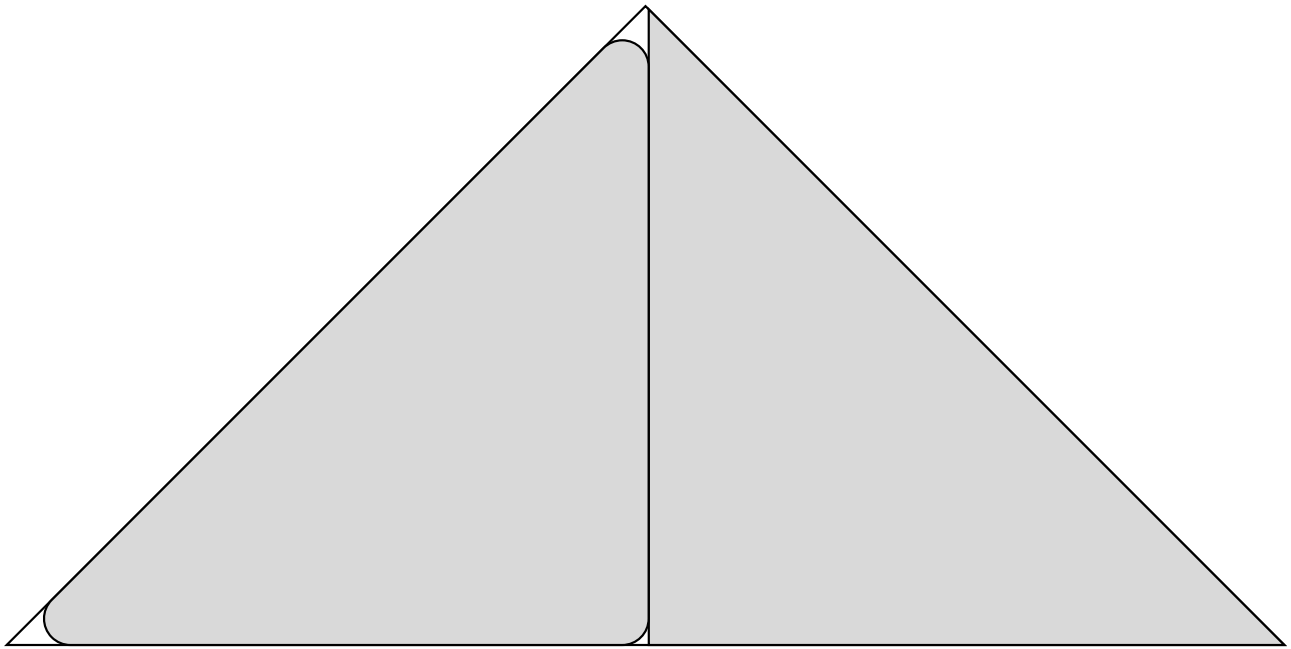

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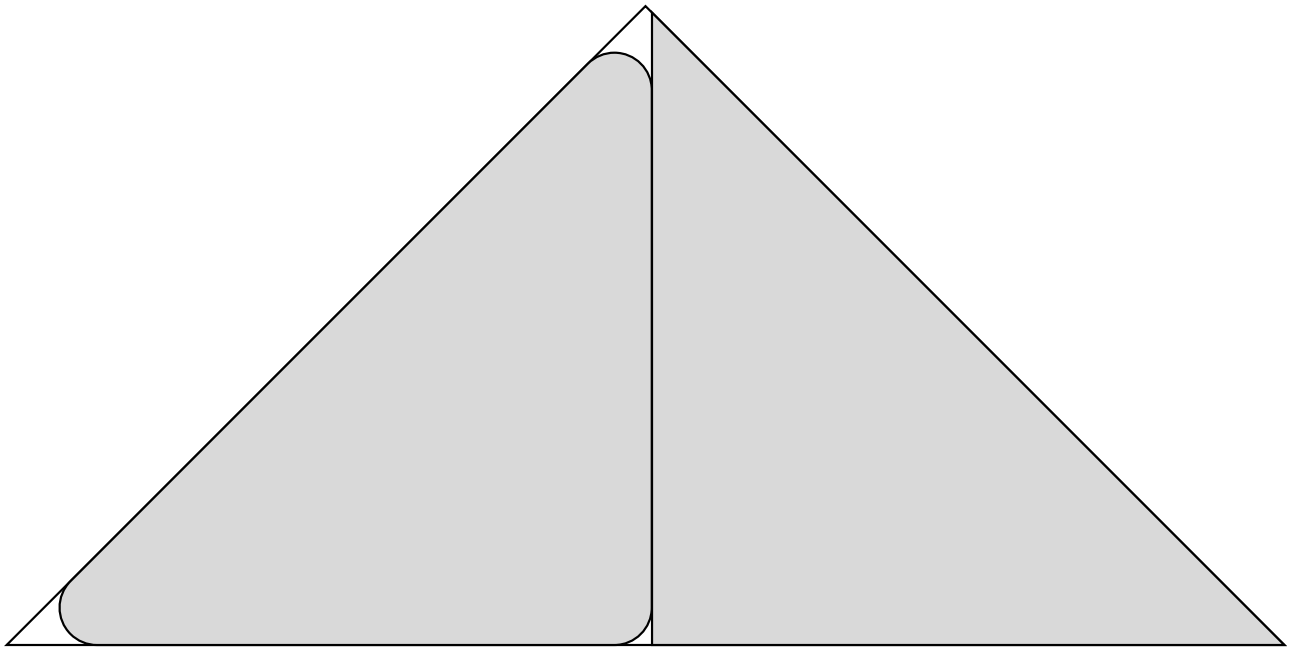

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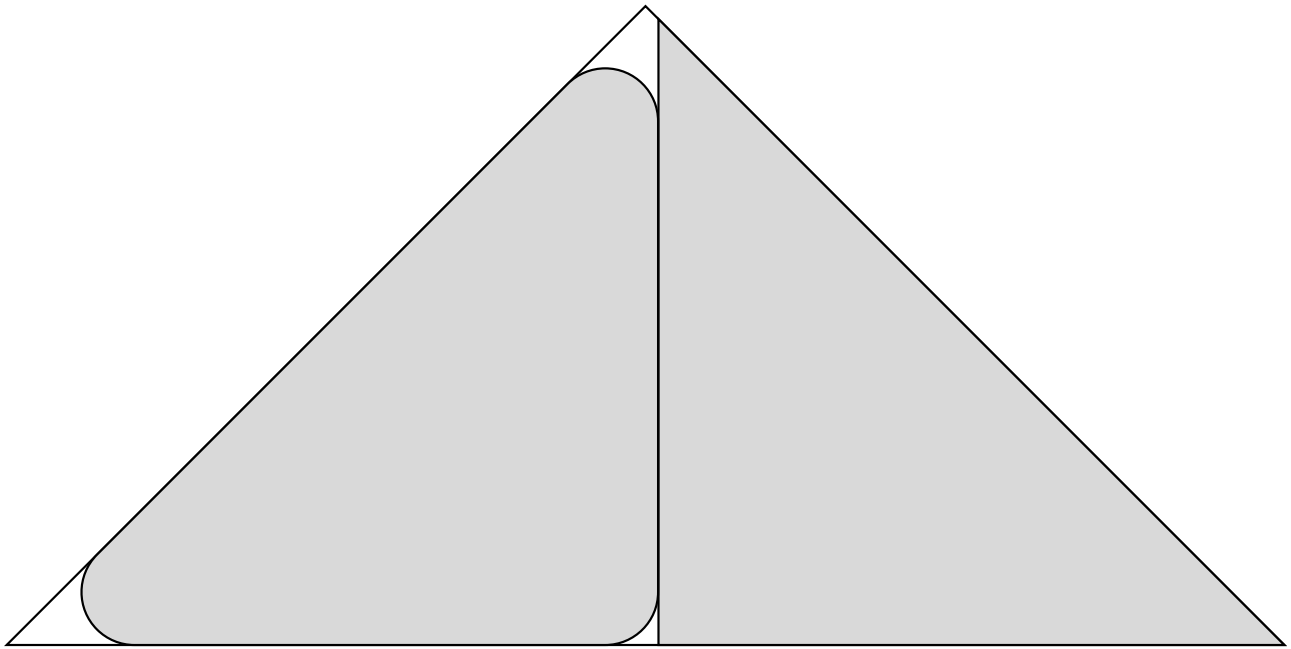

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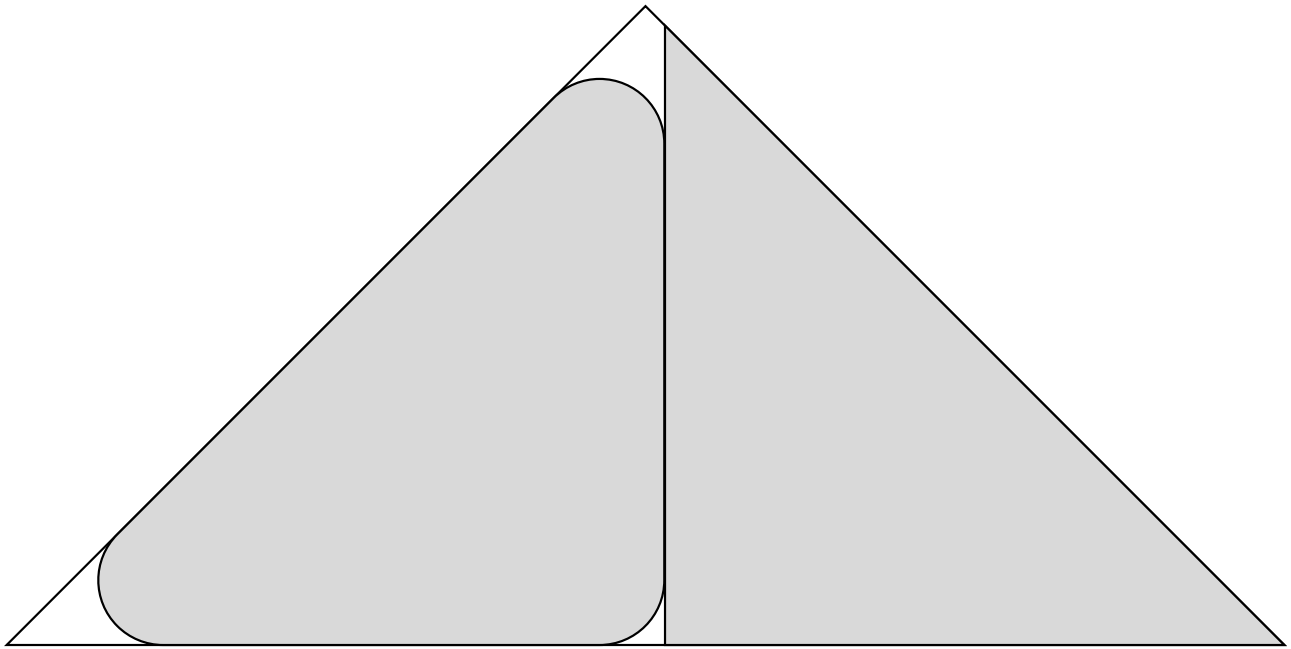

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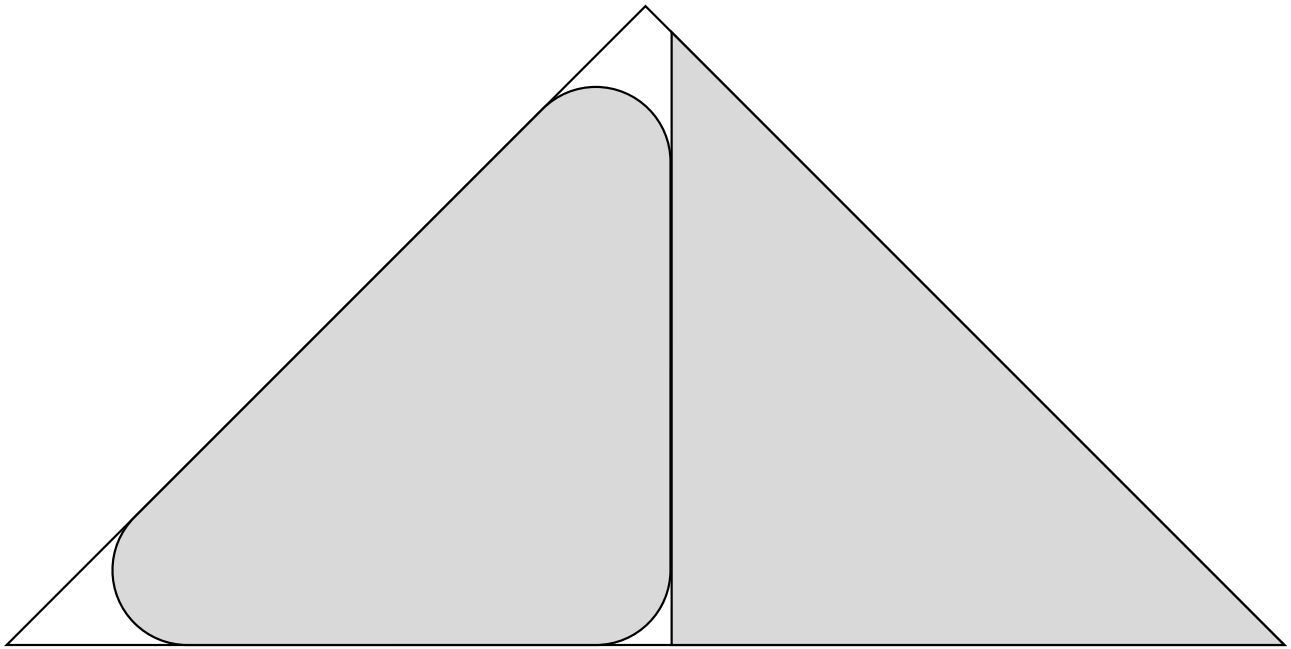

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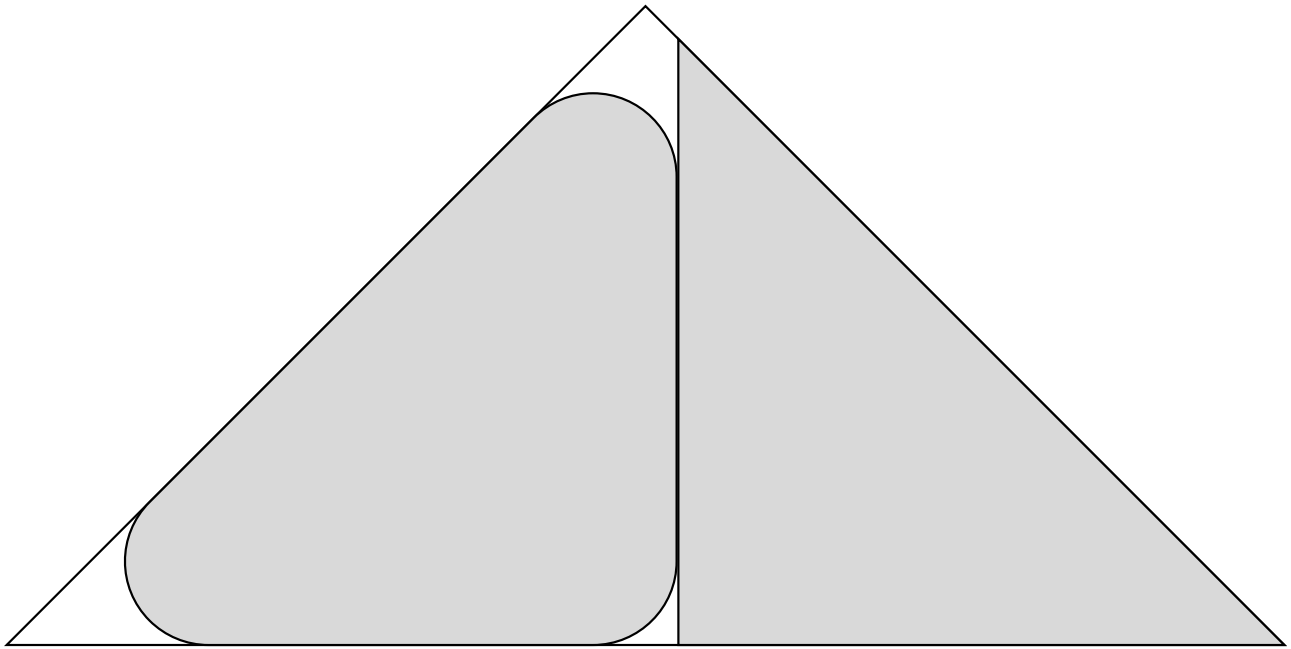

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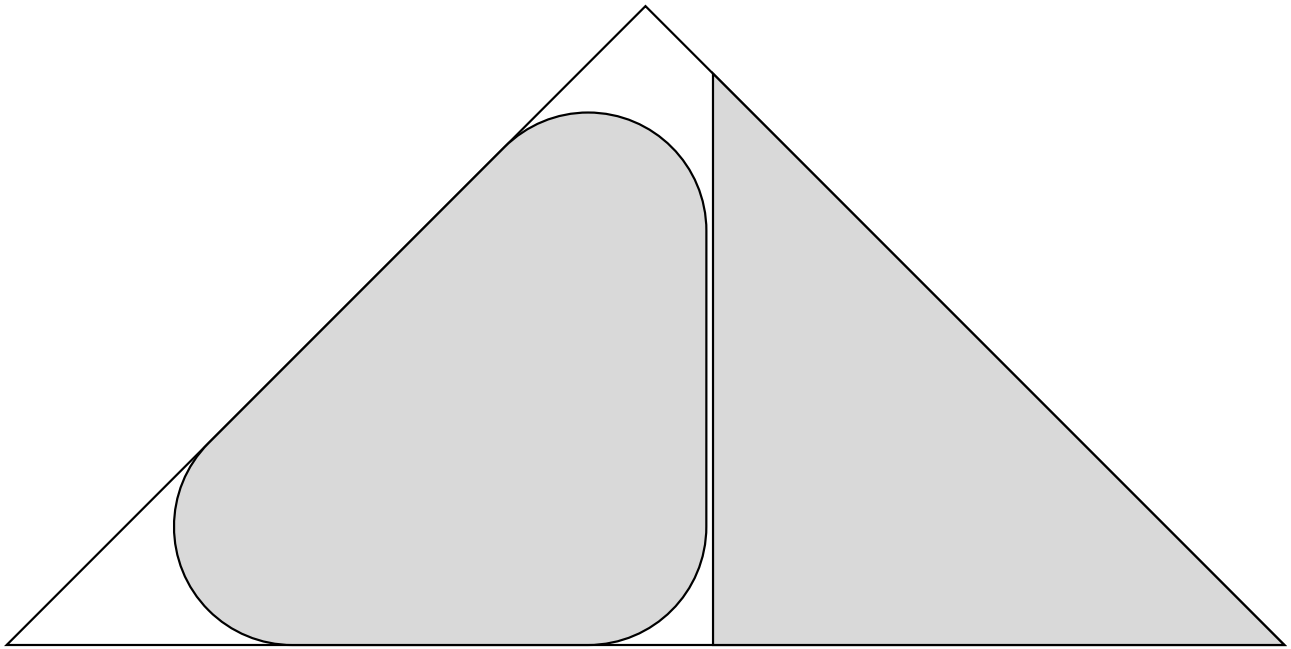

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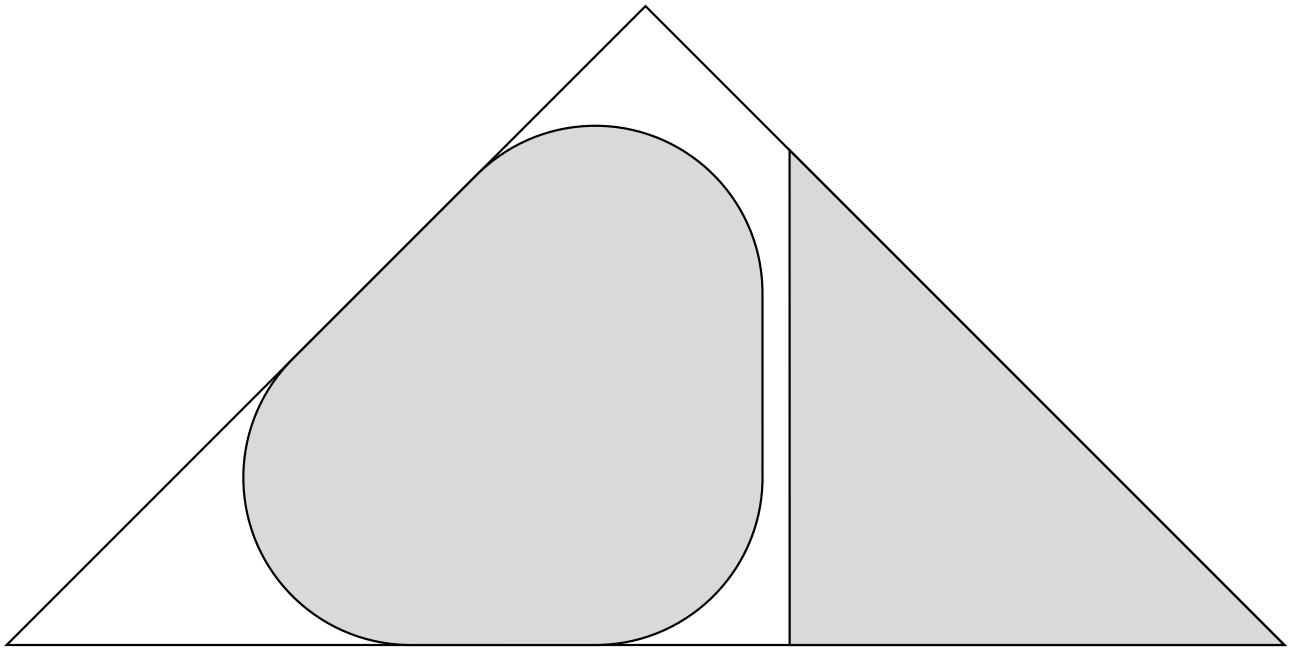

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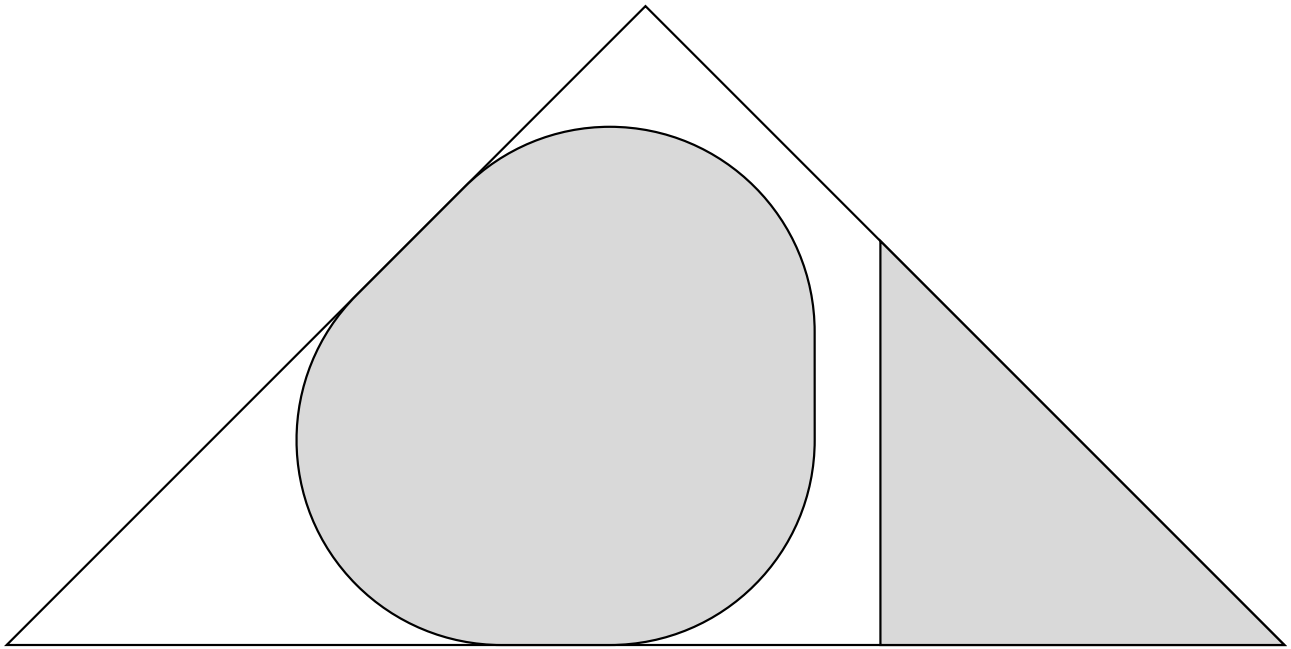

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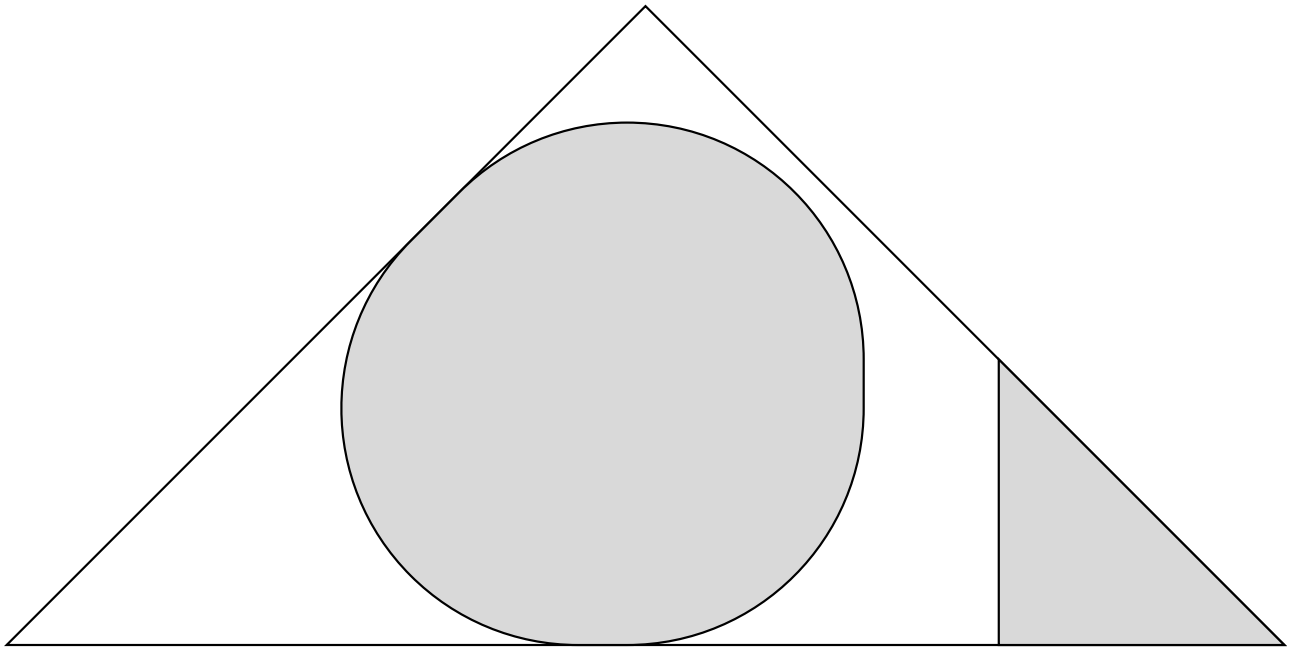

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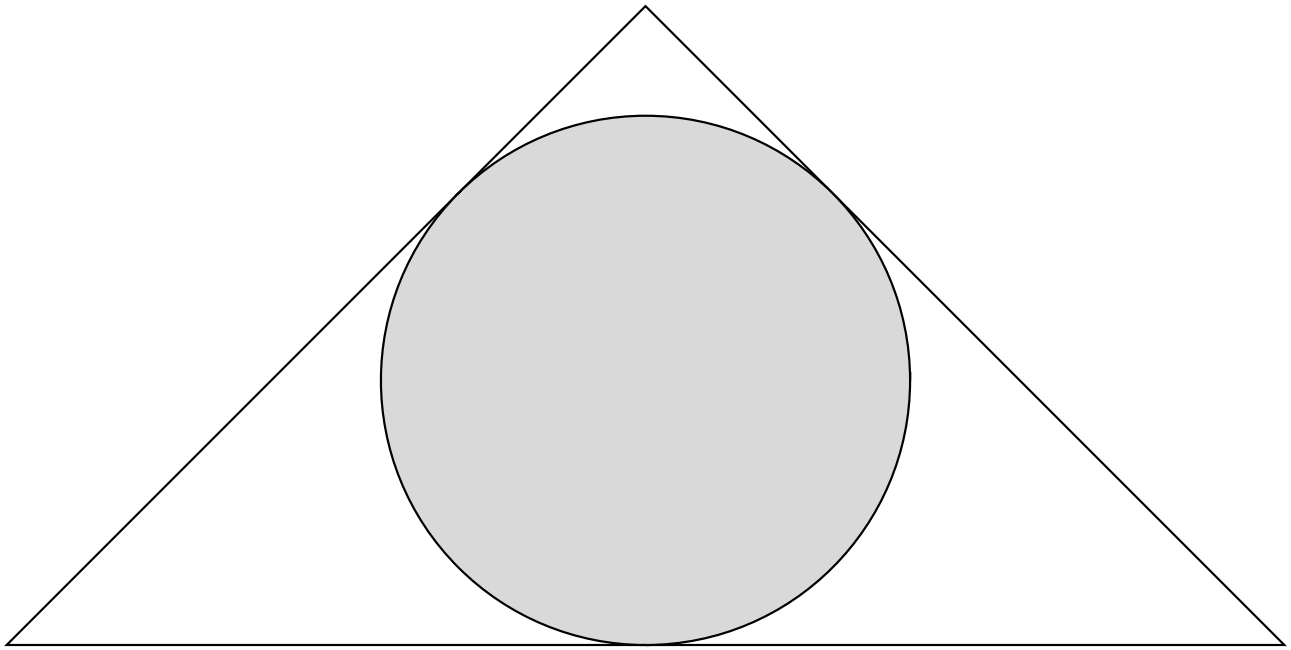

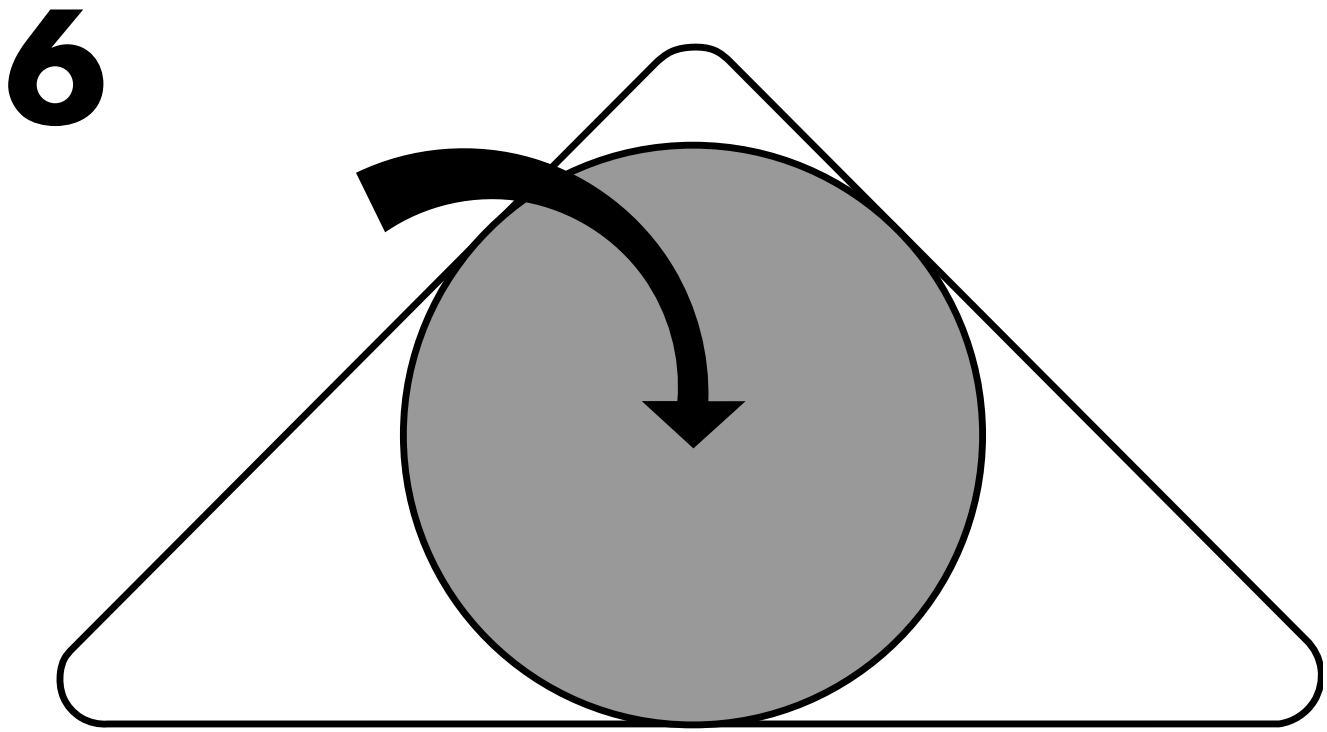
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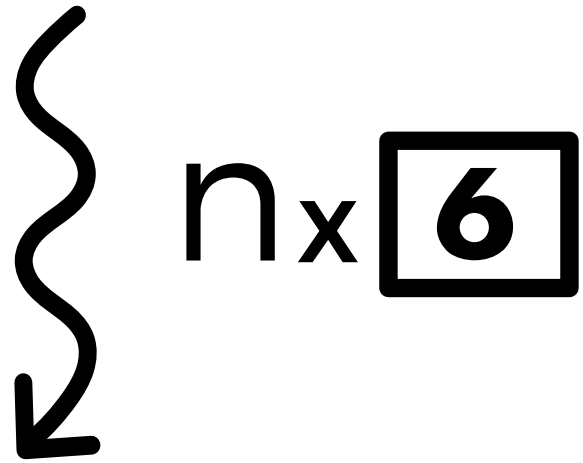
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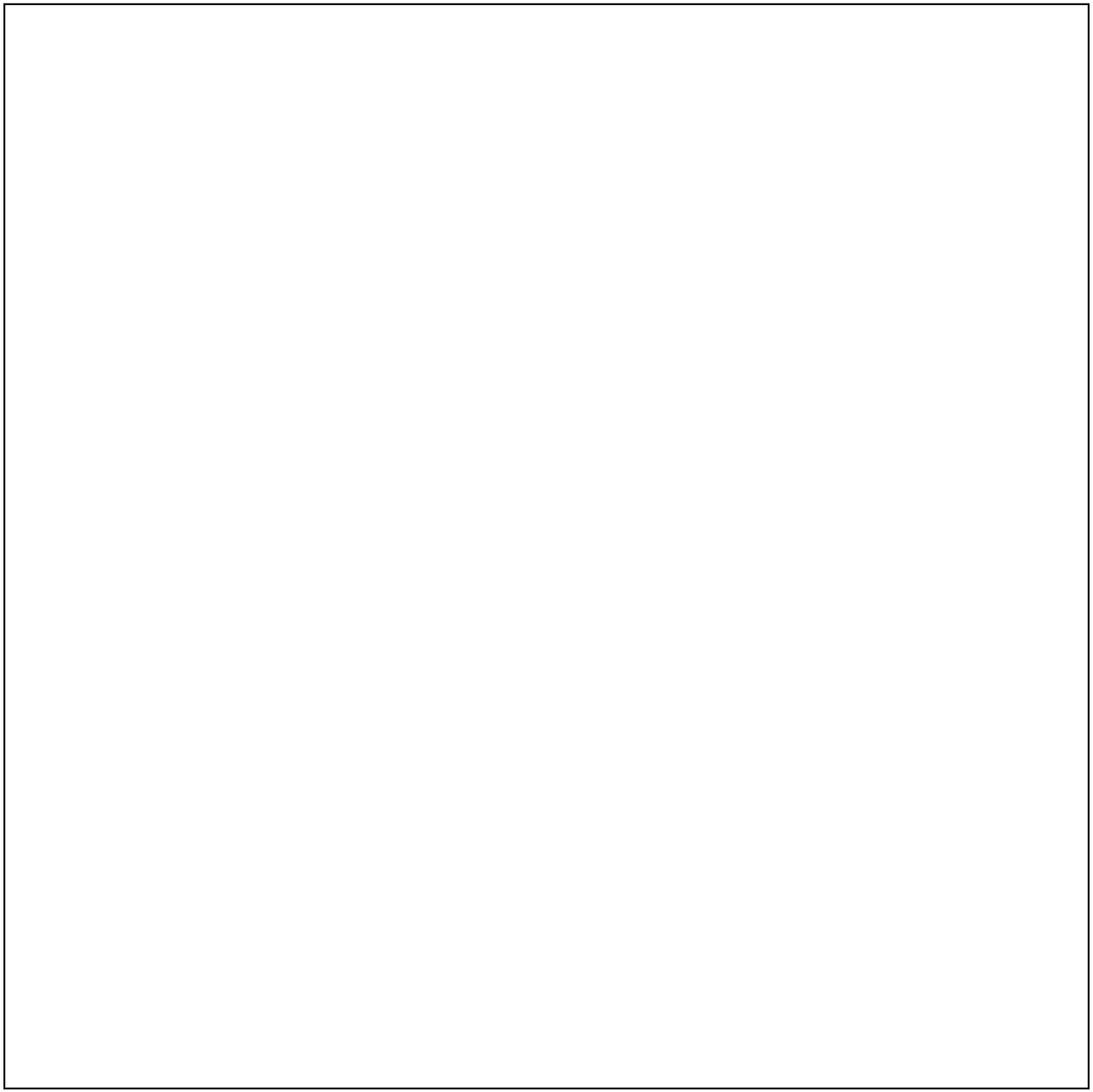

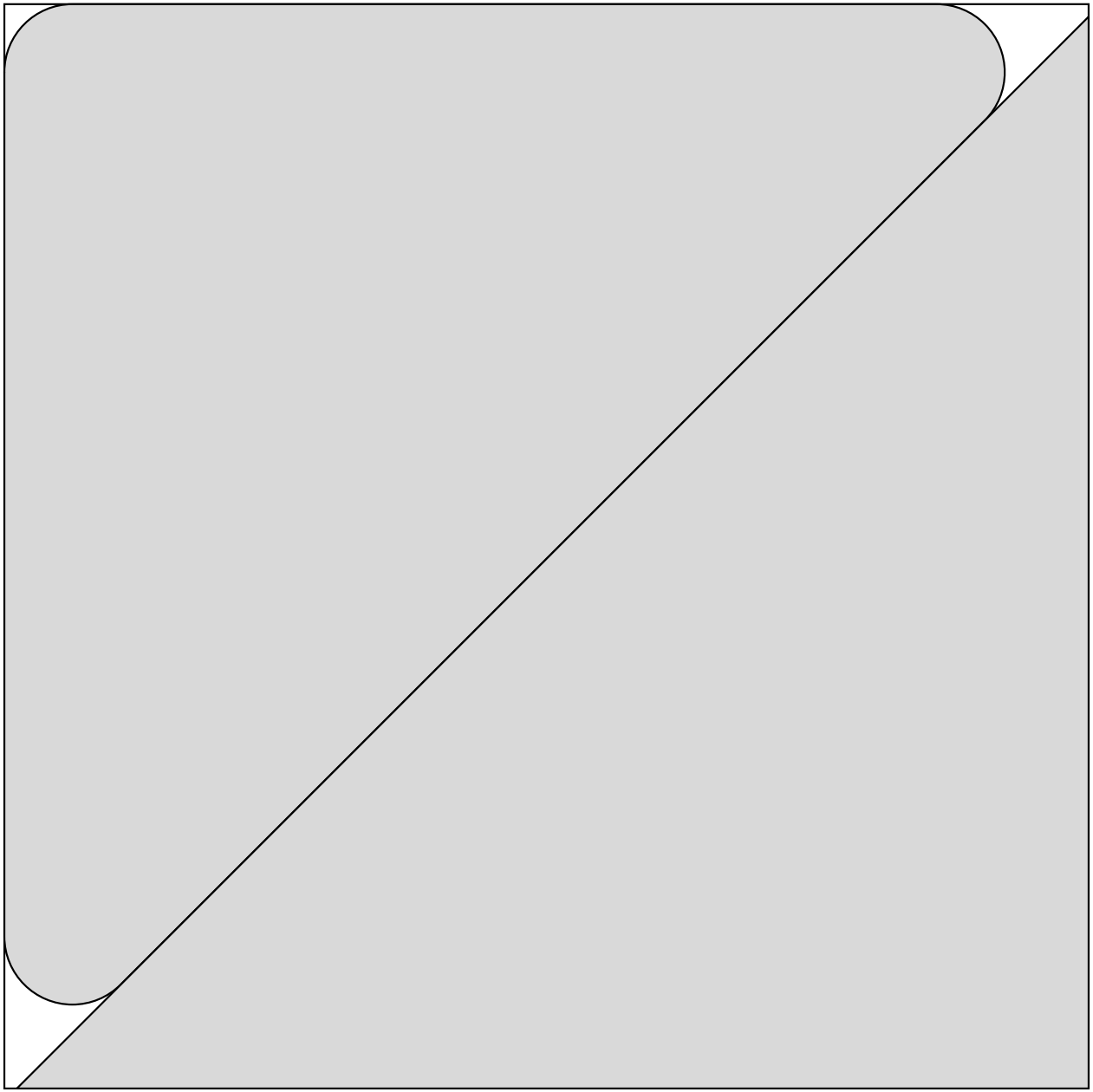

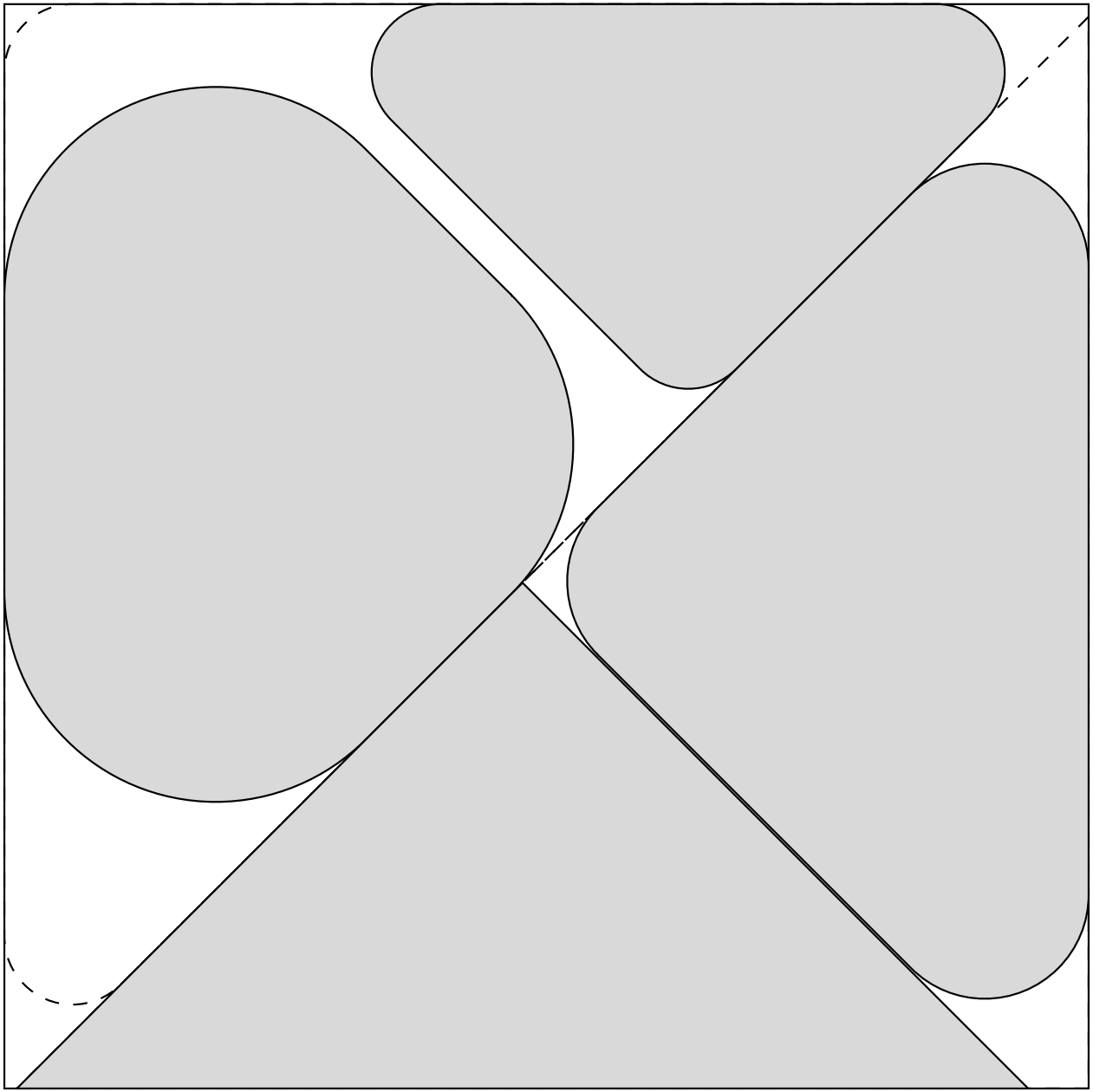

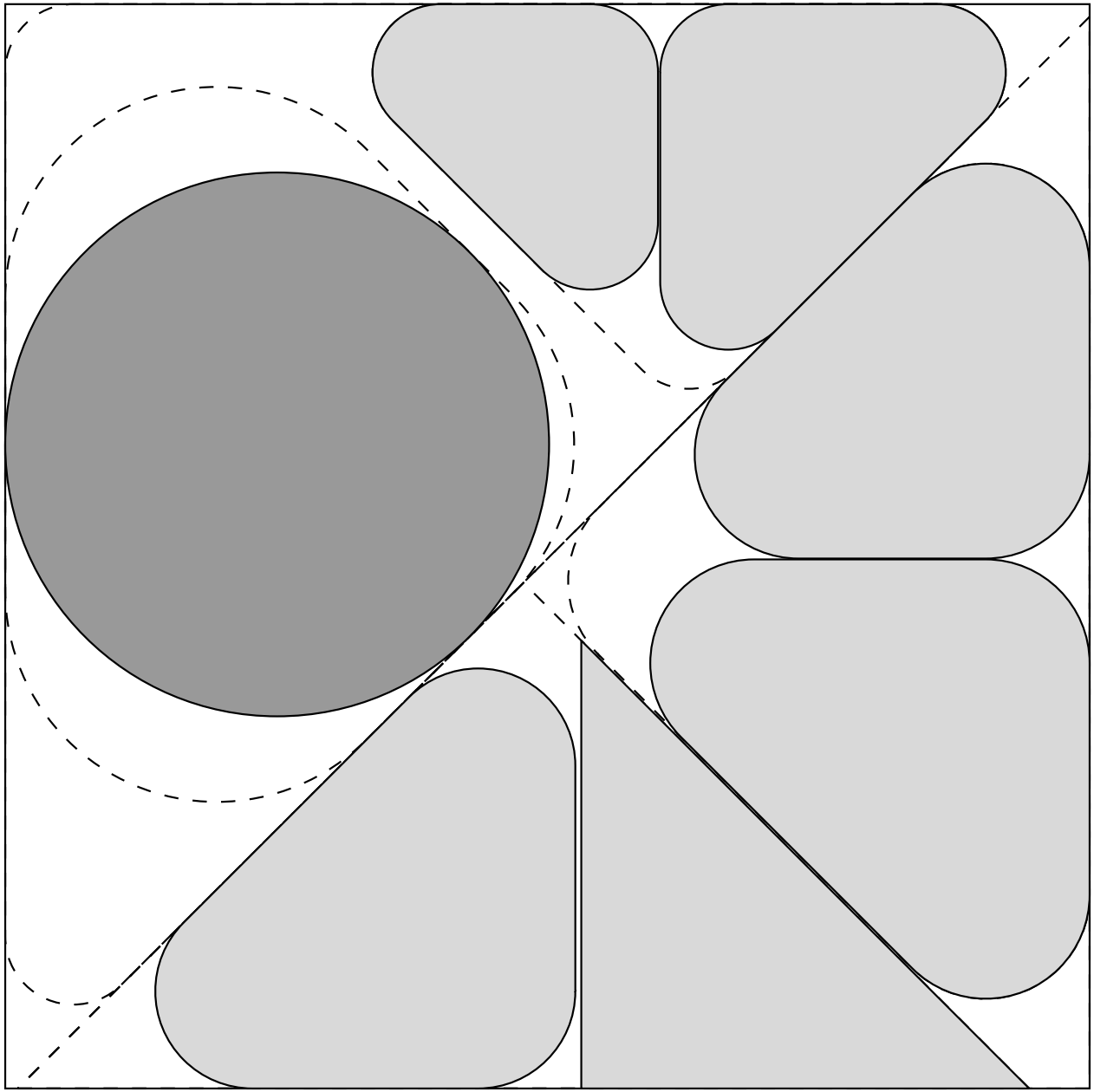

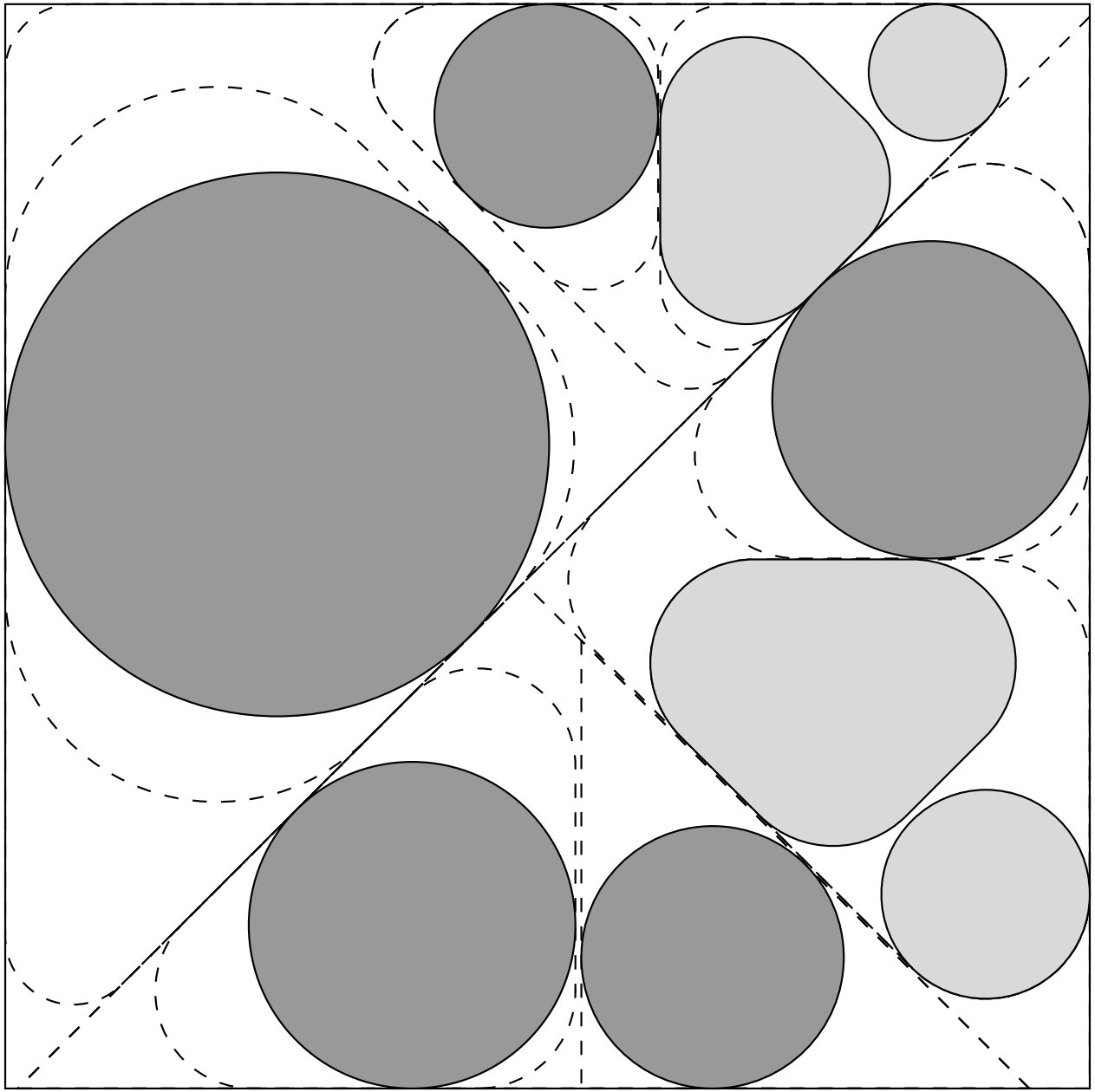

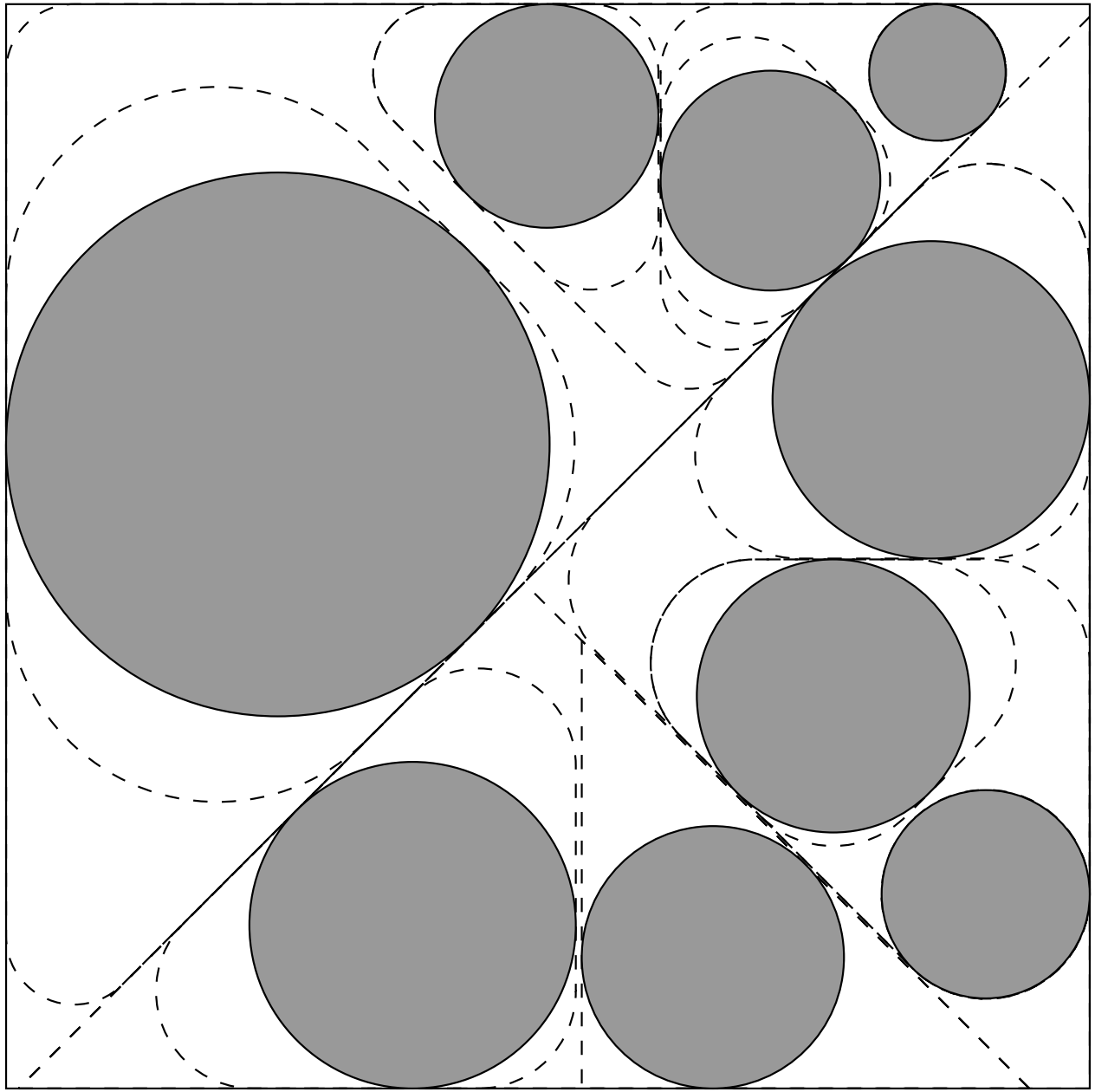

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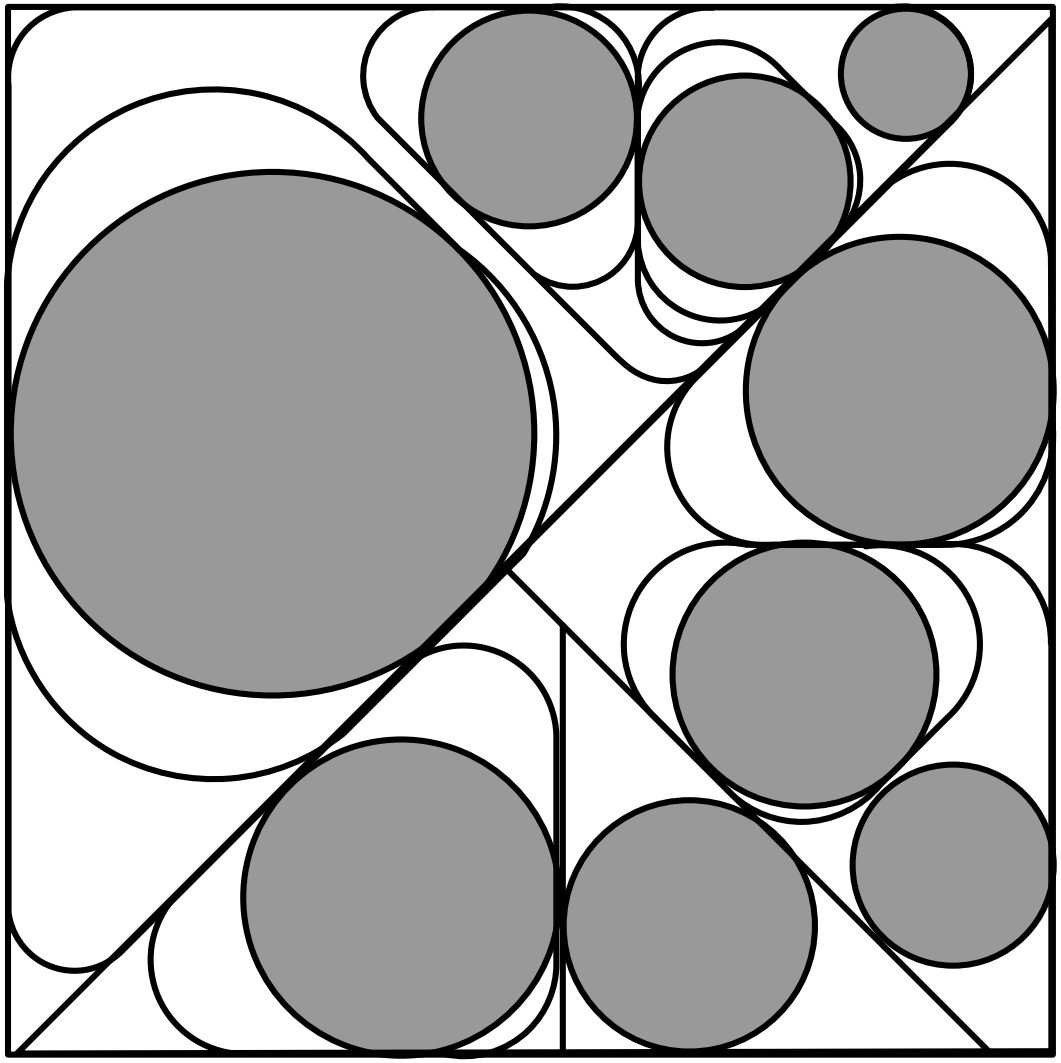

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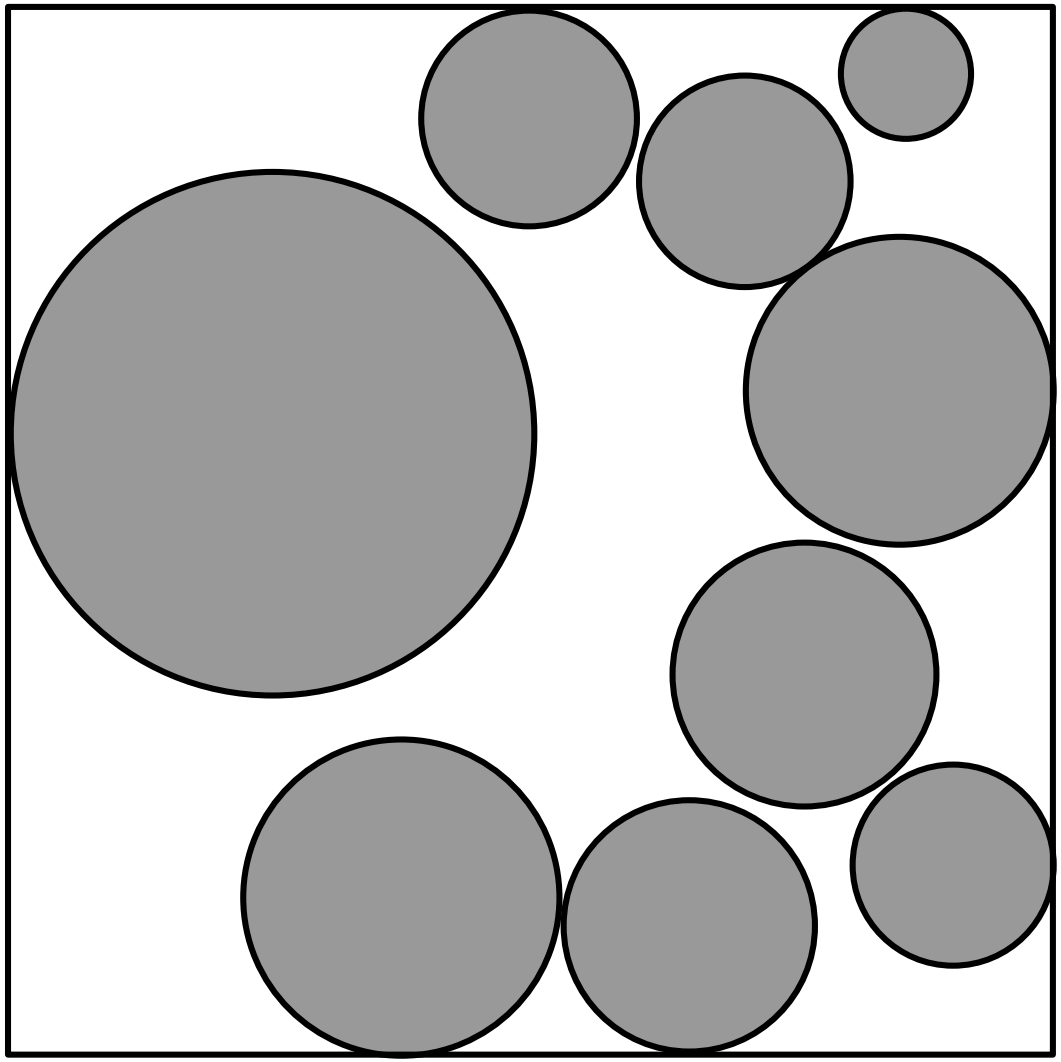

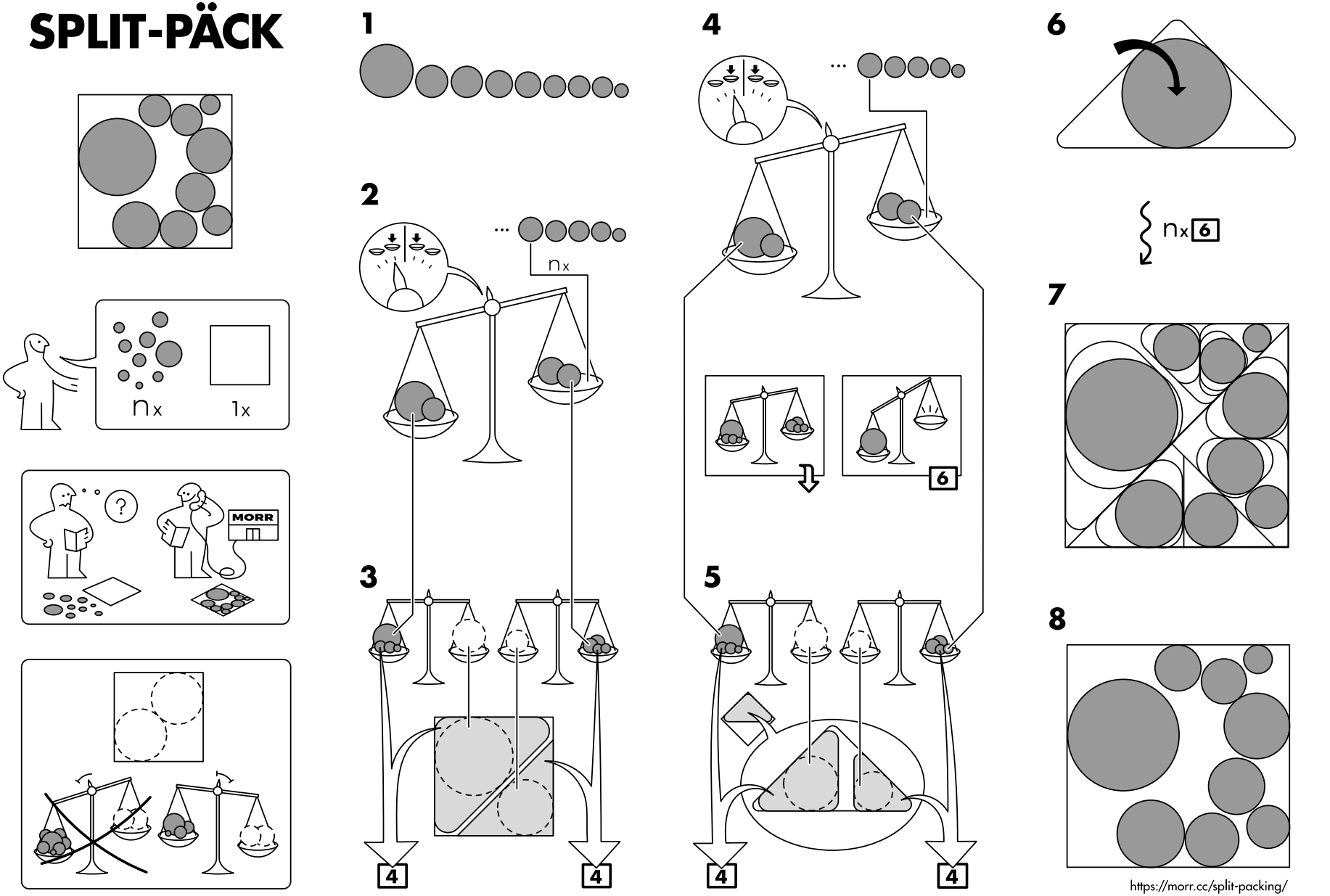
SPLIT-PÄCK
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https://morr.cc/split-packing/

Analysis

Two perspectives:

Analysis

Two perspectives:

Deciding packability

A tight sufficient density condition:
Every instance with up to critical density d can be packed!

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The ratio between the approximated and the optimal container area is at most $1/d$.

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- At most $\mathcal{O}(n^2)$ numeric operations
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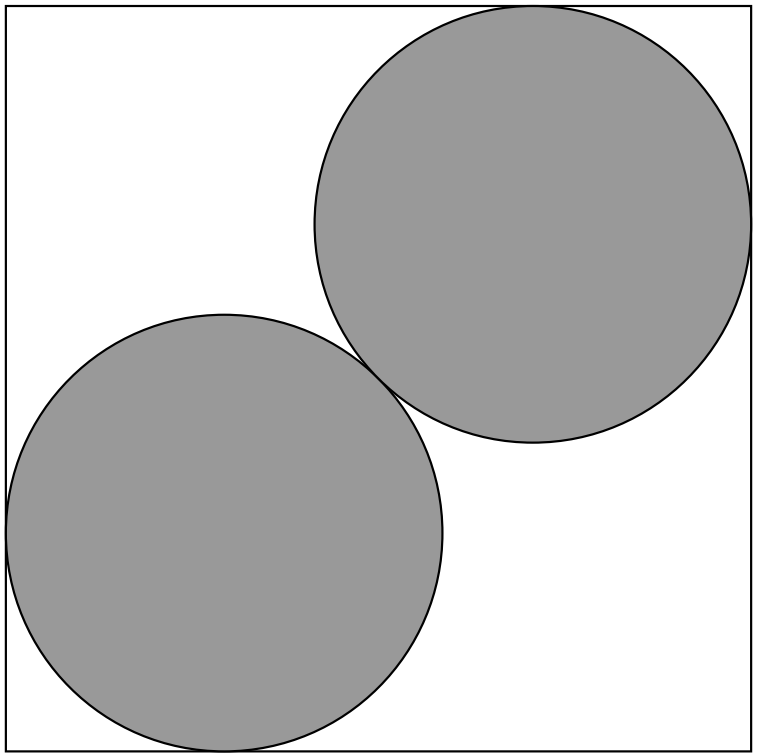
Runtime:

- At most $\mathcal{O}(n^2)$ numeric operations
  - Worst-case greedy split: $n + (n-1) + (n-2) + \cdots + 1$ operations
- Exactly $3n - 2$ geometric constructions
  - Full binary recursion tree with n leaf nodes

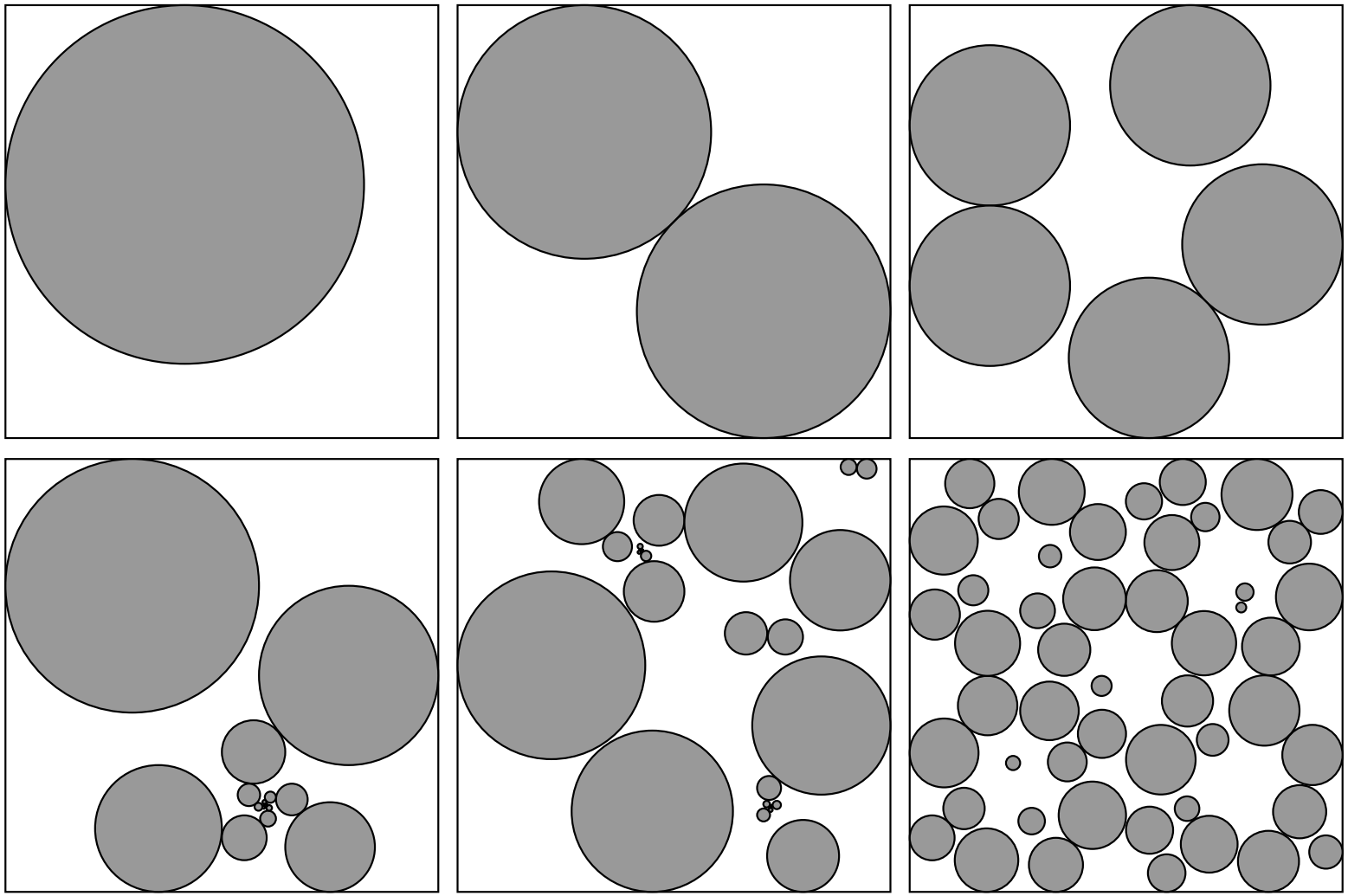
Circles in a square



Critical density: $\frac{\pi}{3+2\sqrt{2}} \approx 53.90\%$

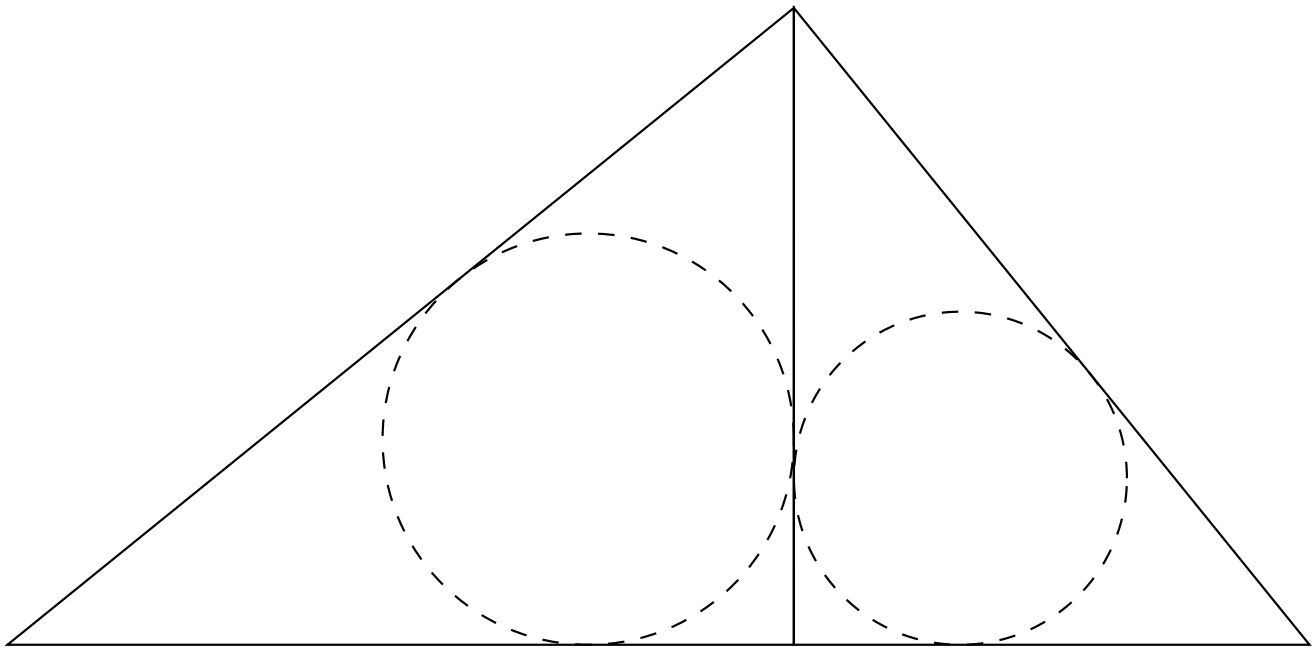
Approximation factor: $\frac{3+2\sqrt{2}}{\pi} \approx 1.8552$

Circles in a square: Examples

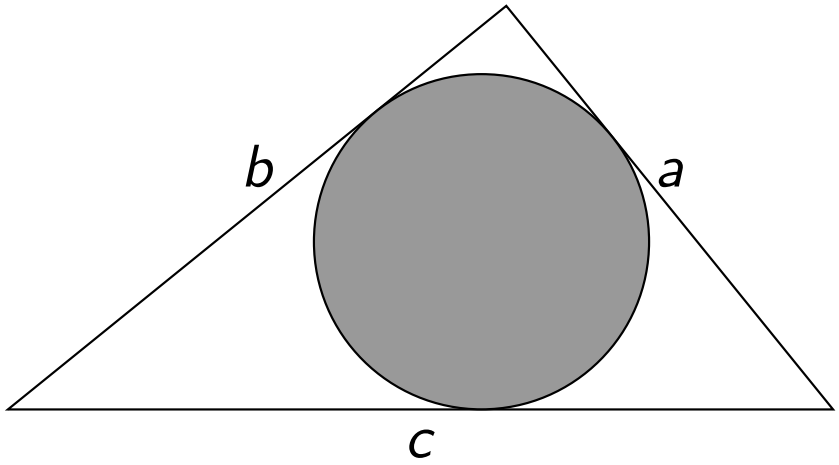


Outline

Splitting for asymmetric triangles



Circles in a right/obtuse triangle

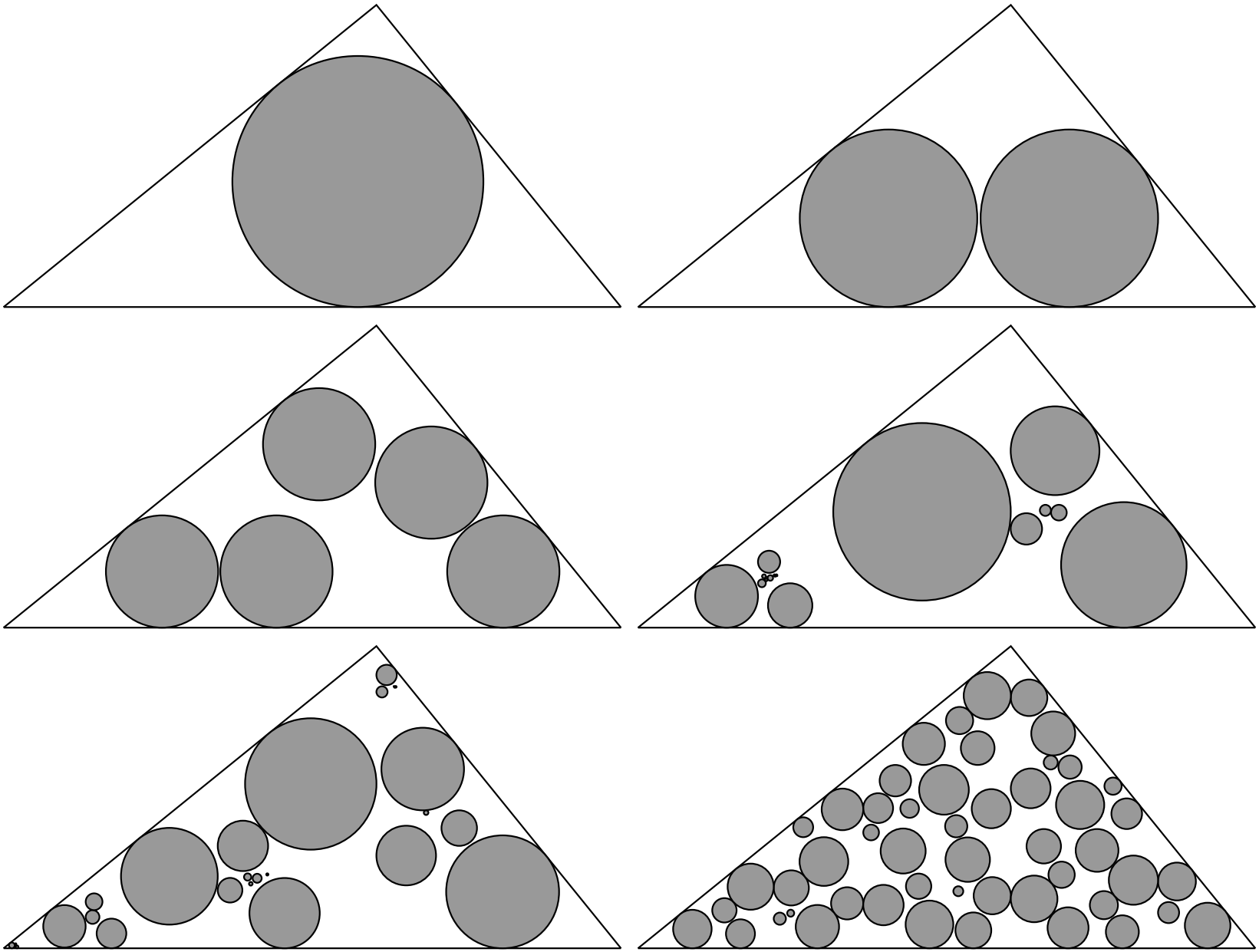


Condition: $a^2 + b^2 \leq c^2$

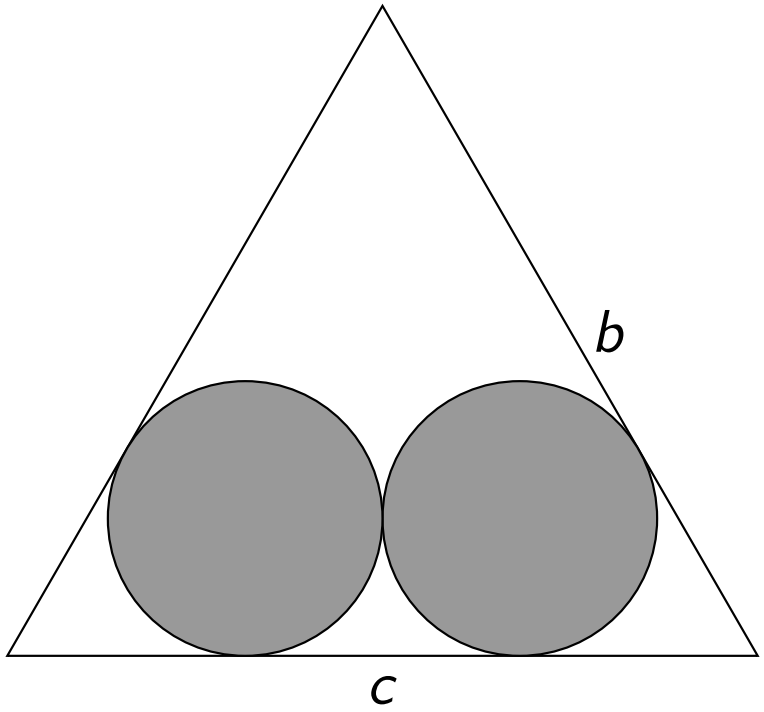
Critical density: $\sqrt{\frac{-(a-b-c)(a+b-c)(a-b+c)}{(a+b+c)^3}}\pi < 53.91\%$

Approximation factor: Larger than 1.8552

Circles in a right/obtuse triangle: Examples



Circles in a thick isosceles triangle

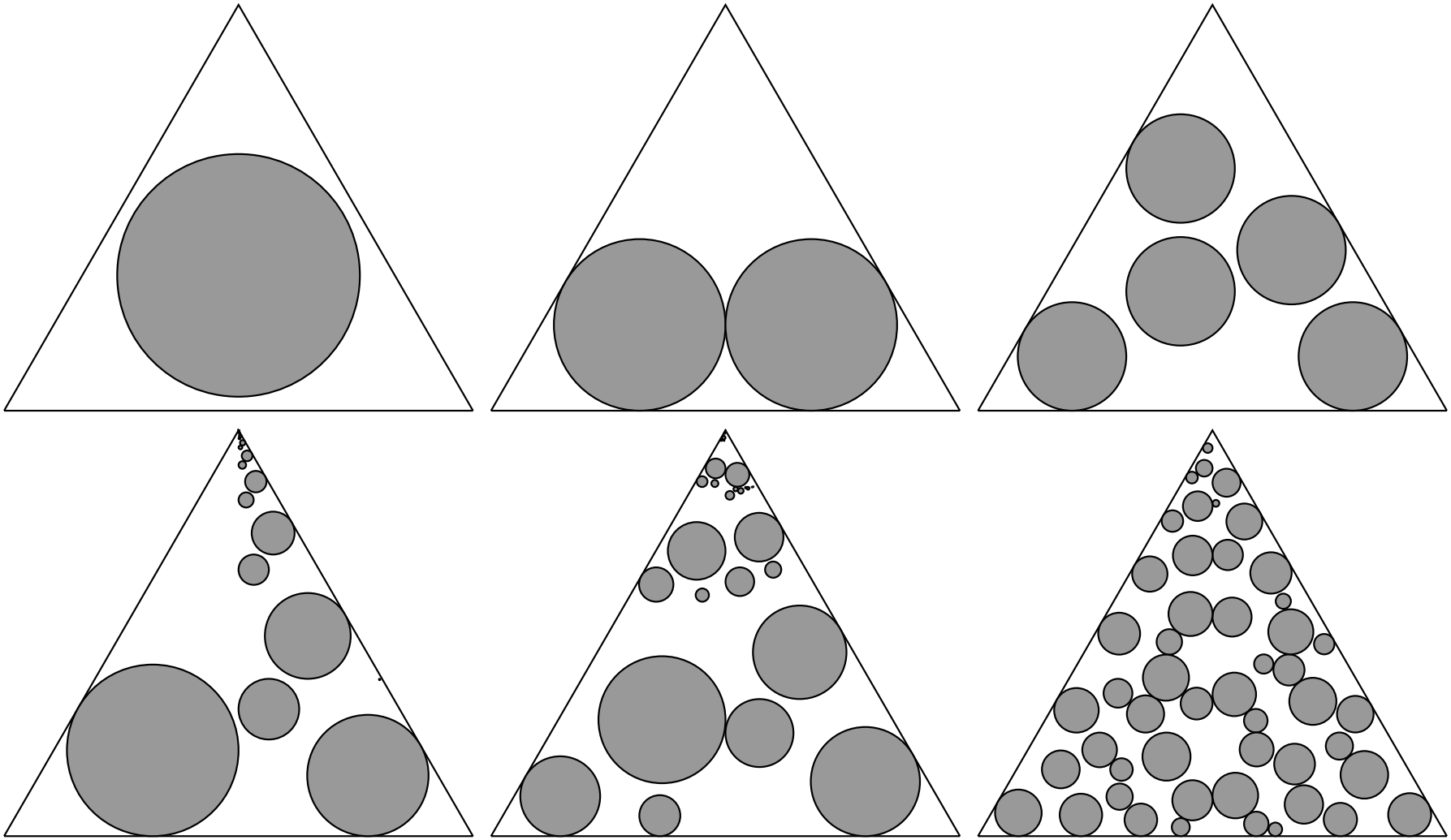


Condition: $\frac{c}{\sqrt{2}} \leq b \leq c$

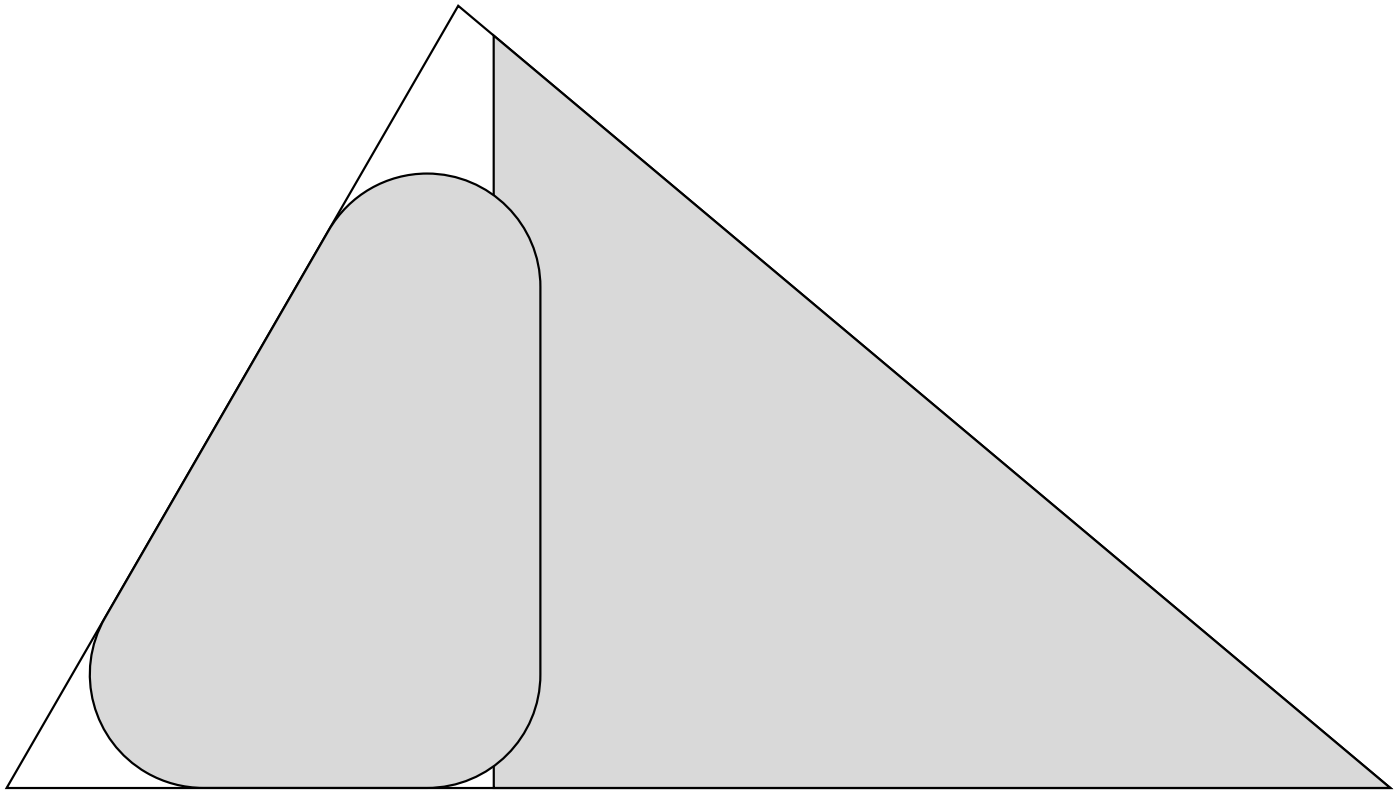
Critical density: $48.60\% < \frac{(c - 2b + \sqrt{4b^2 - c^2})^2 \pi}{2c\sqrt{4b^2 - c^2}} < 53.91\%$

Approximation factor: Between 1.8552 and 2.0576

Circles in a thick isosceles triangle: Examples



The problem with acute triangles



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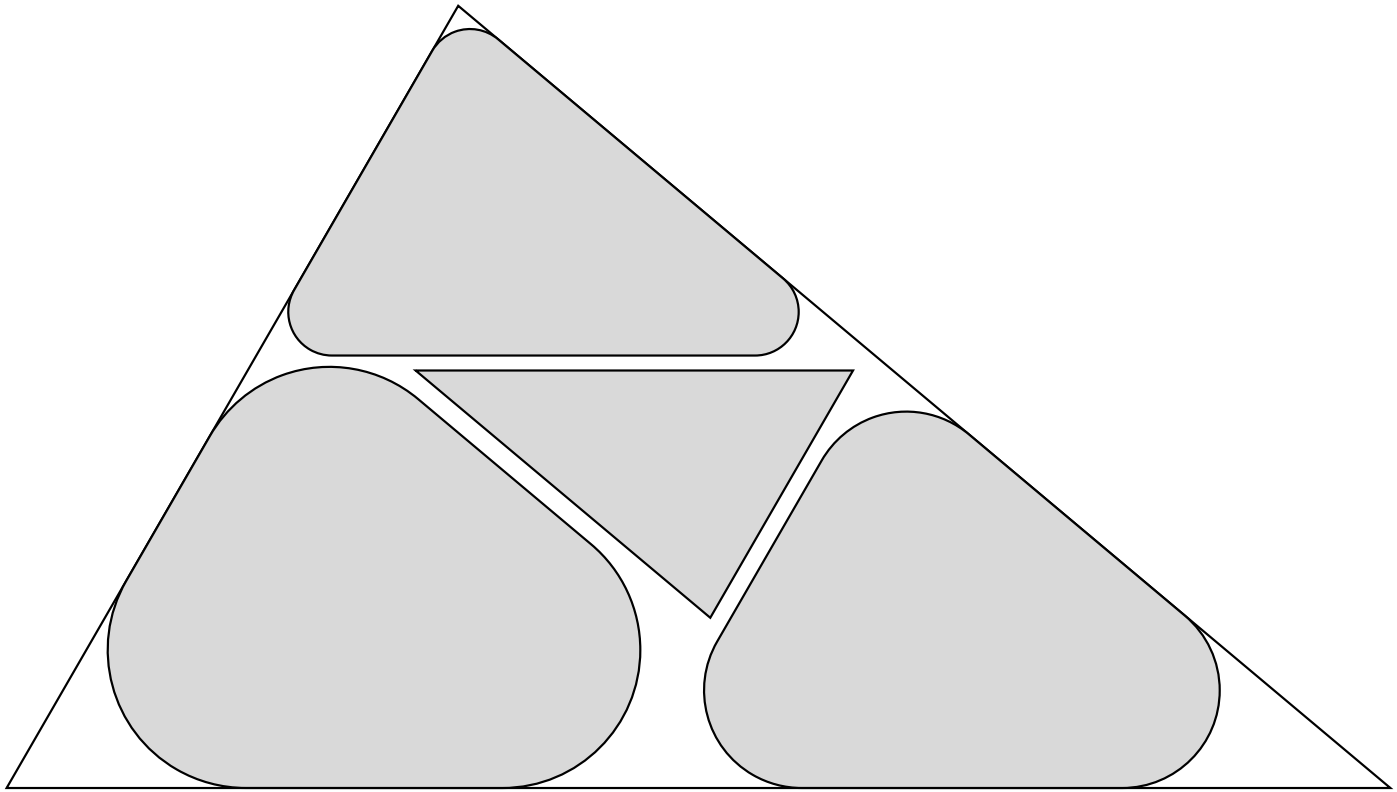

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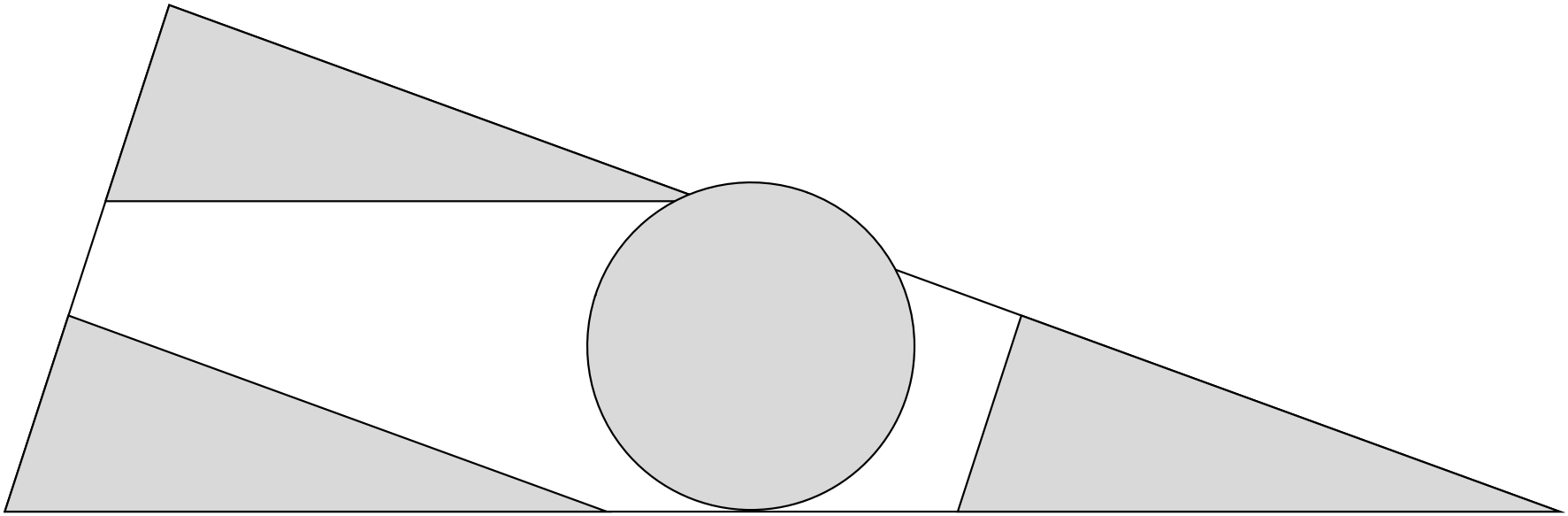
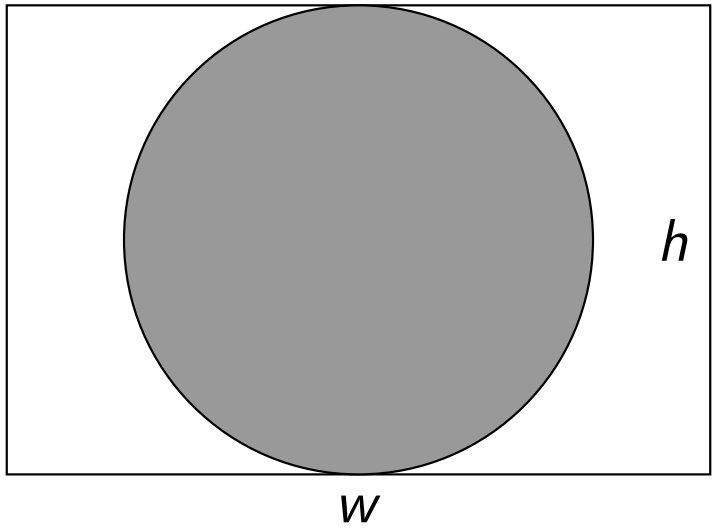


Circles in a long rectangle

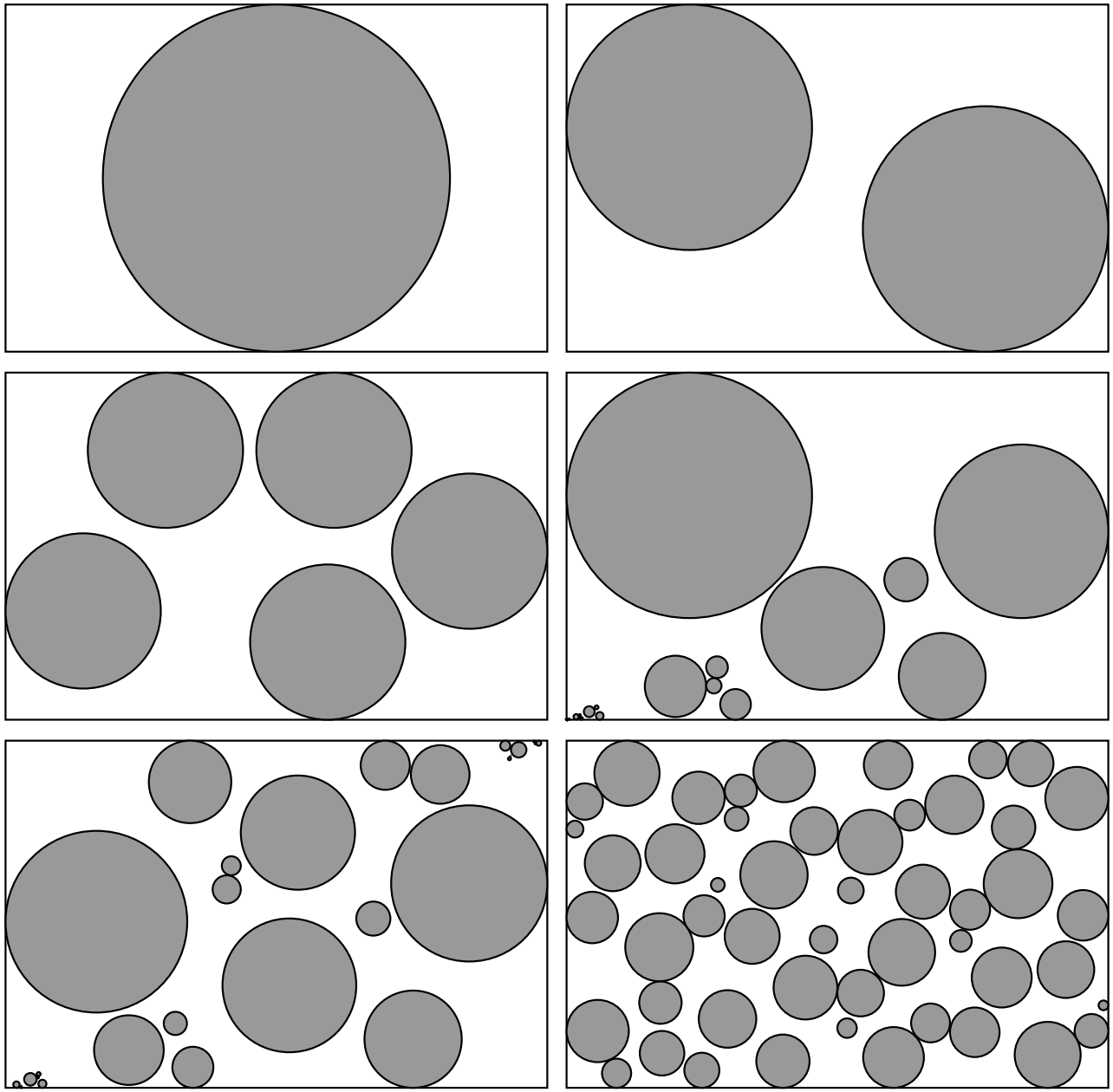


Condition: $w \geq \frac{2+3\sqrt{2}}{4} h \approx 1.5607h$

Critical density: $\frac{\pi h}{4w} < 50.33\%$

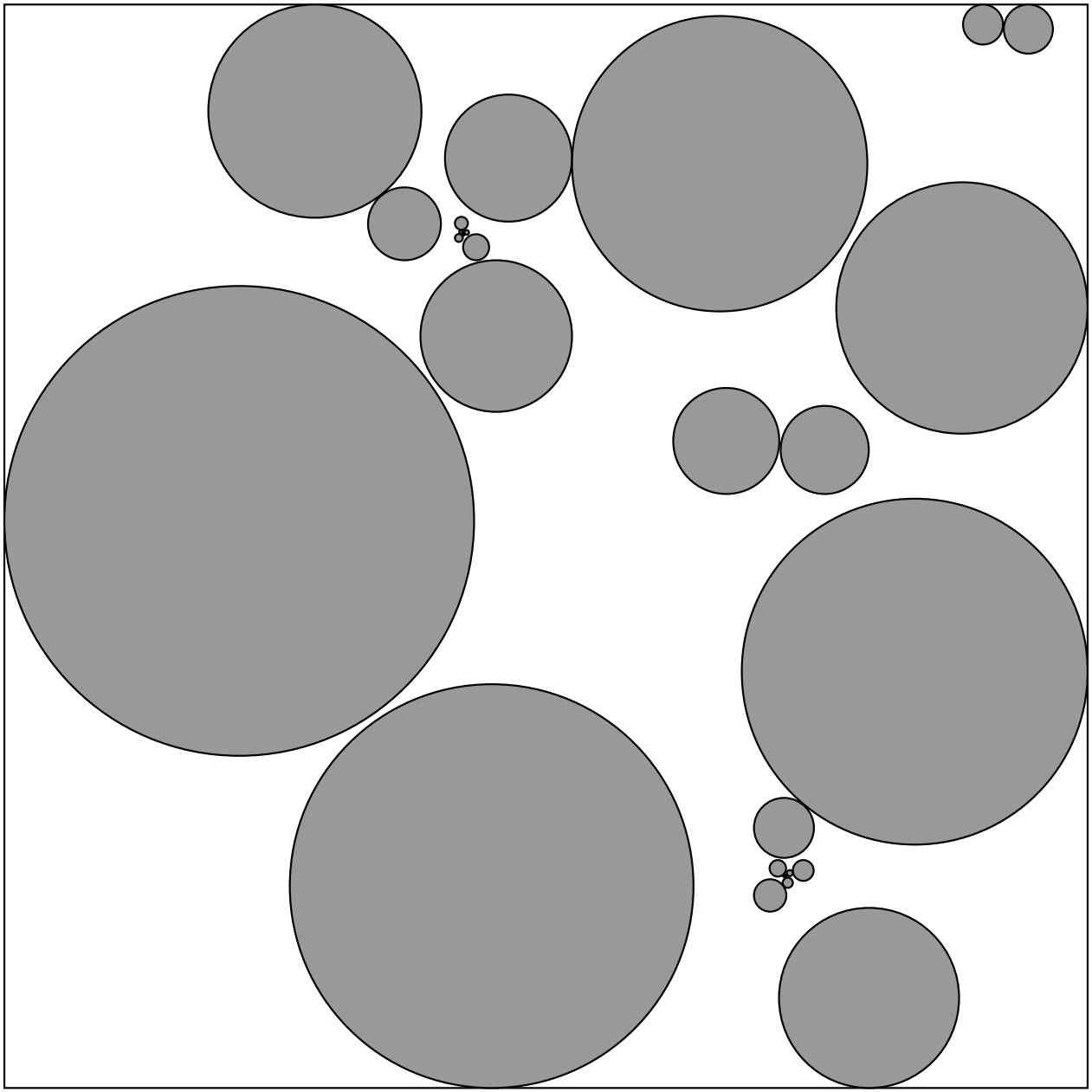
Approximation factor: $\frac{4w}{\pi h} > 1.9870$

Circles in a long rectangle: Examples



Outline

Other shapes in a square?



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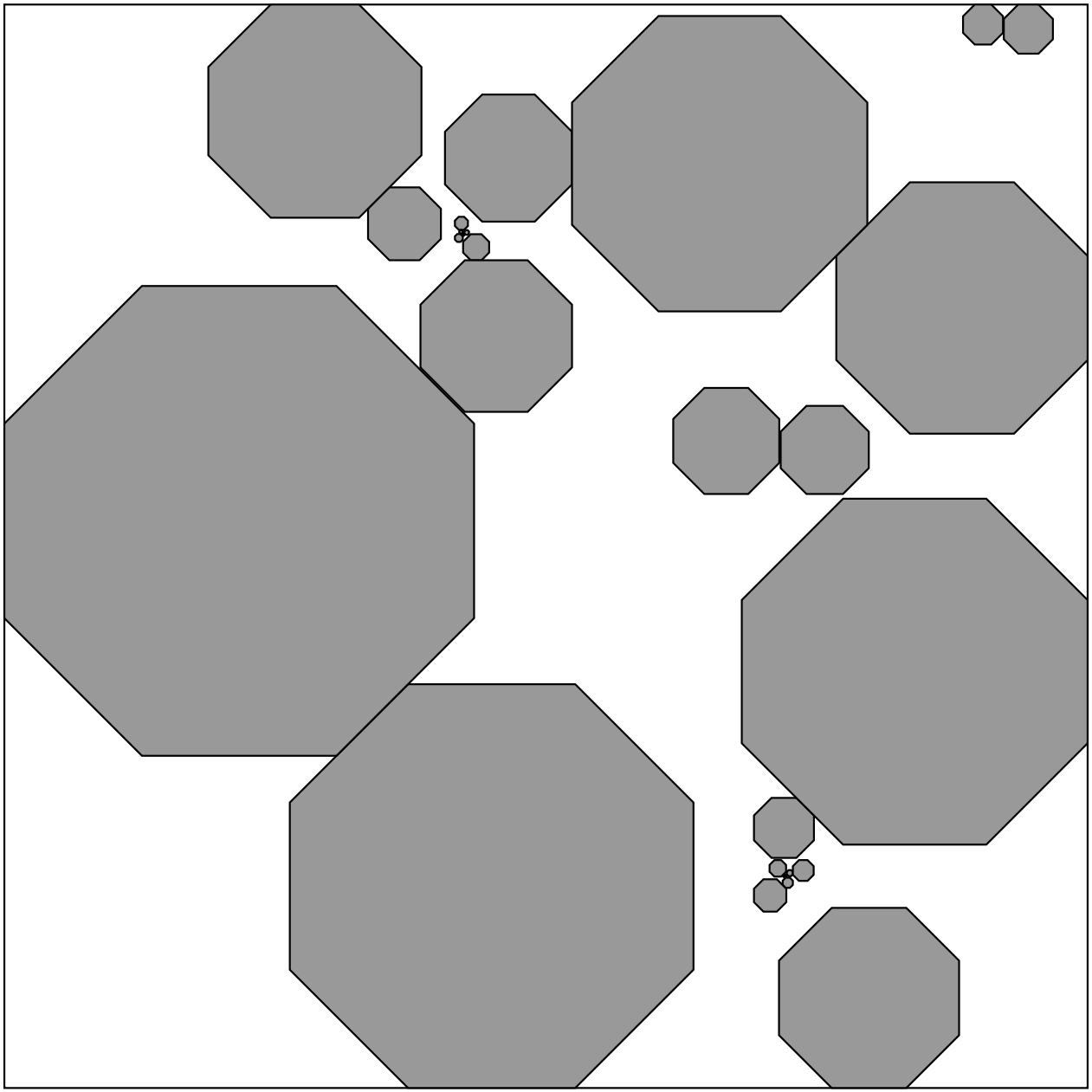
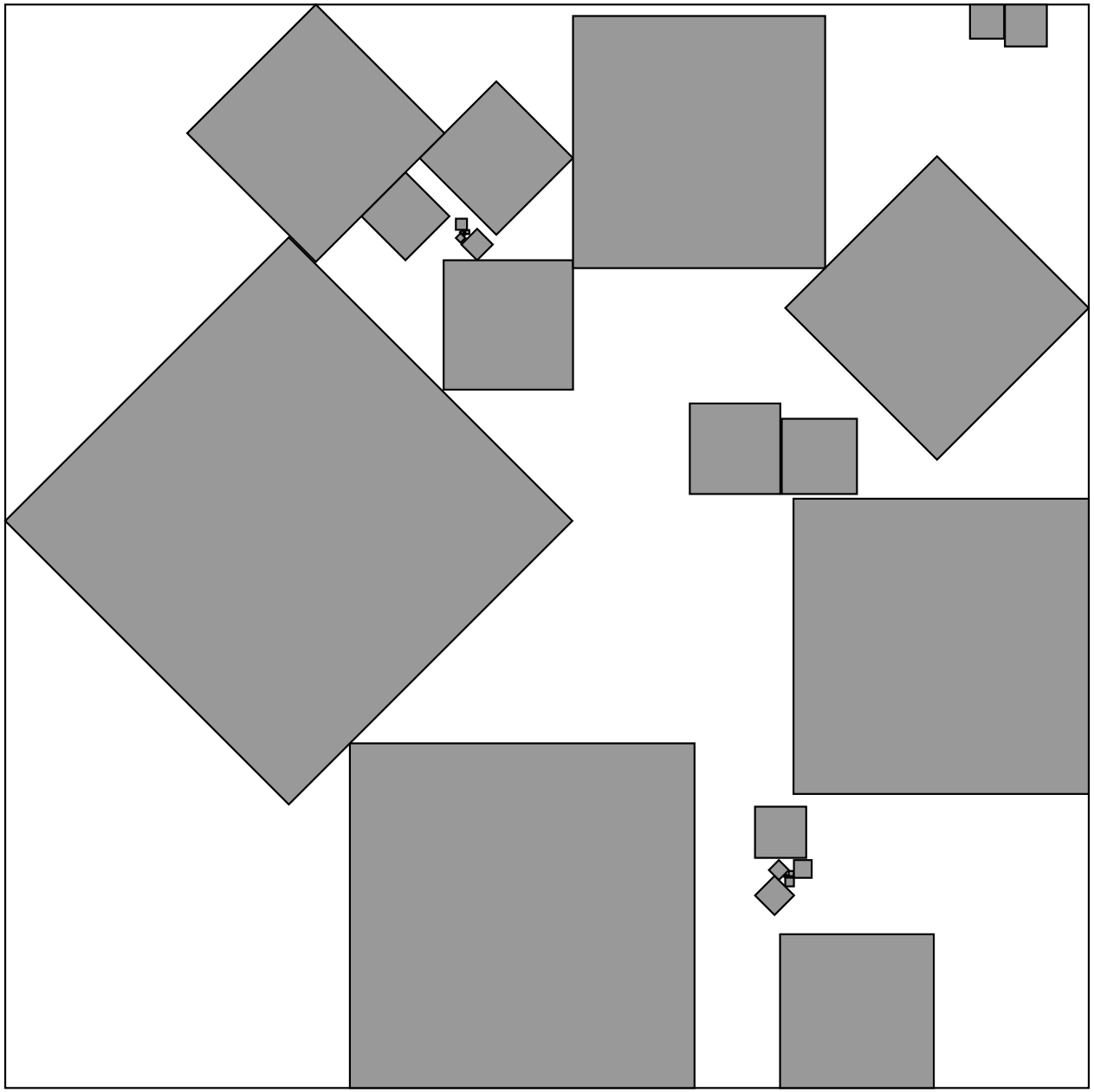
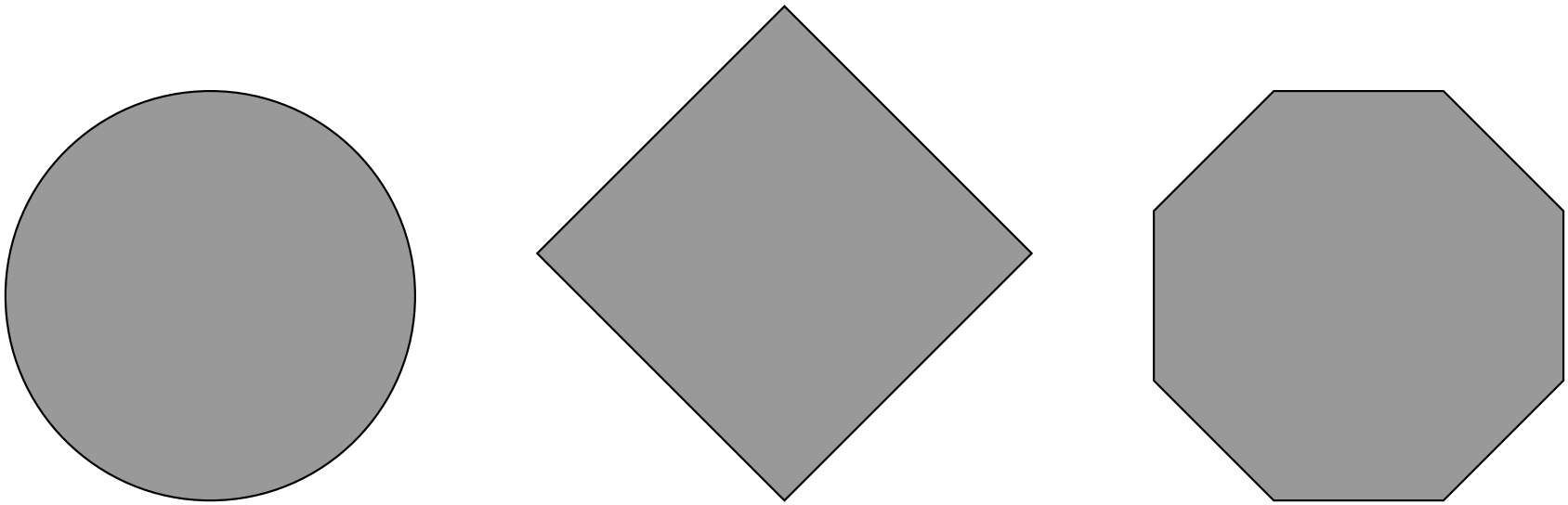


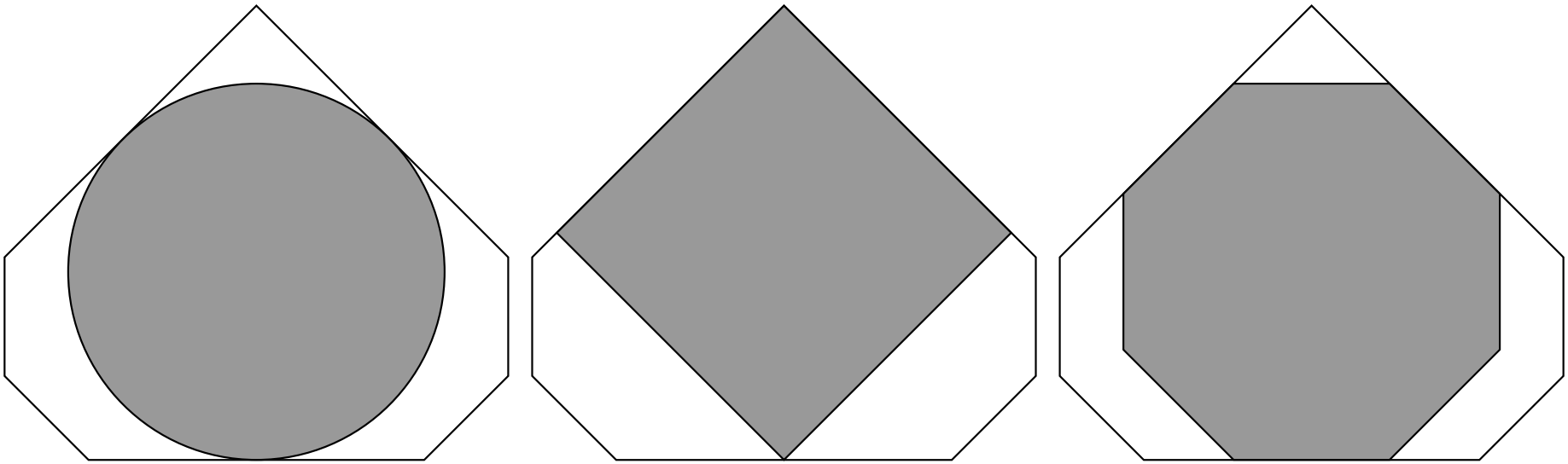
Other shapes in a square?



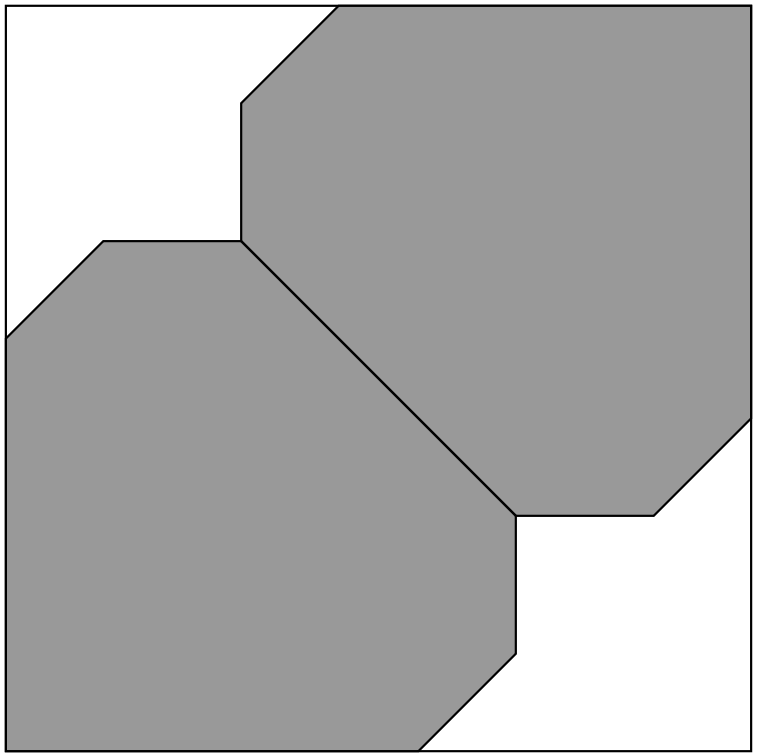
Rubies!



Rubies!



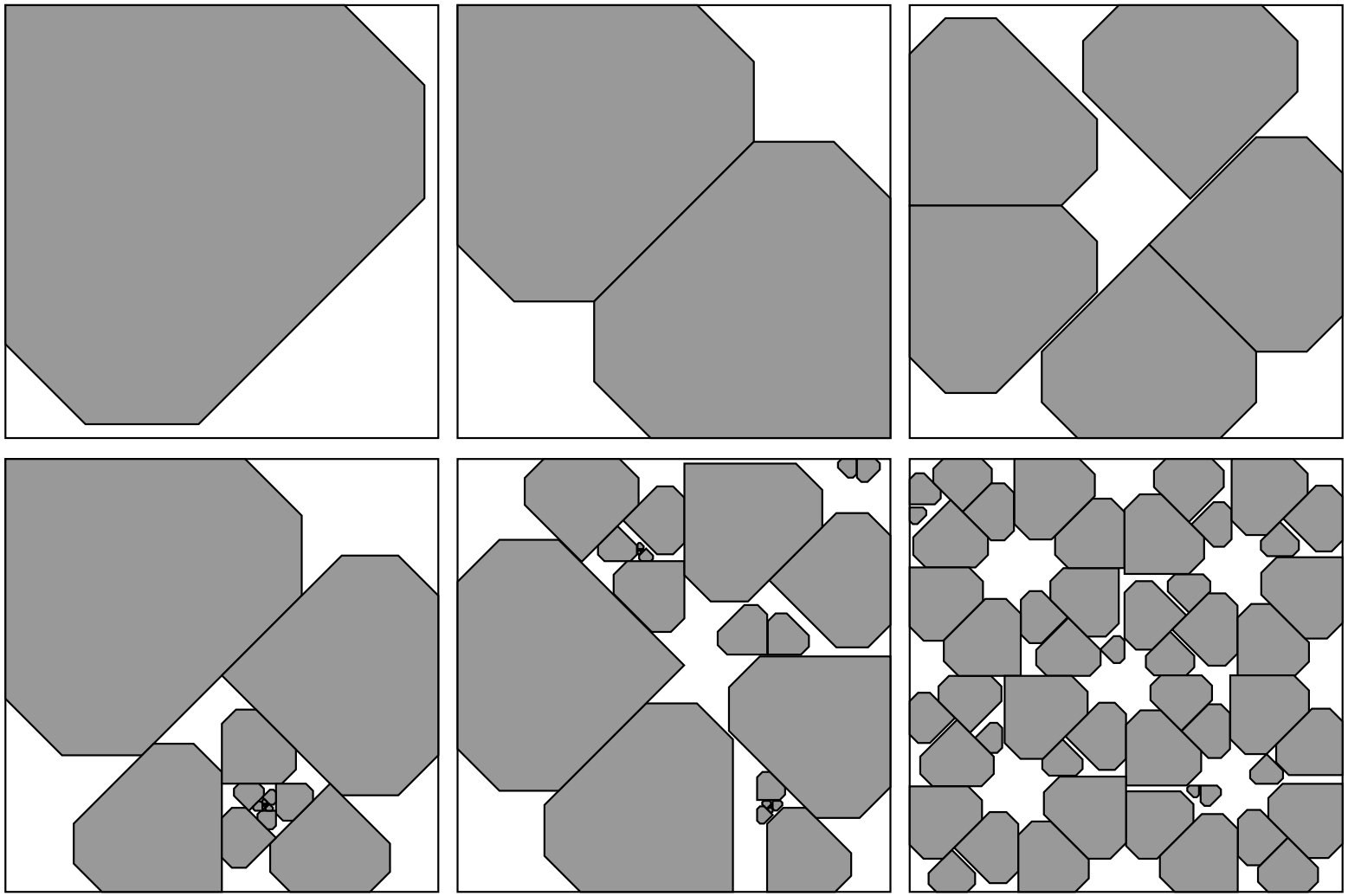
Rubies in a square



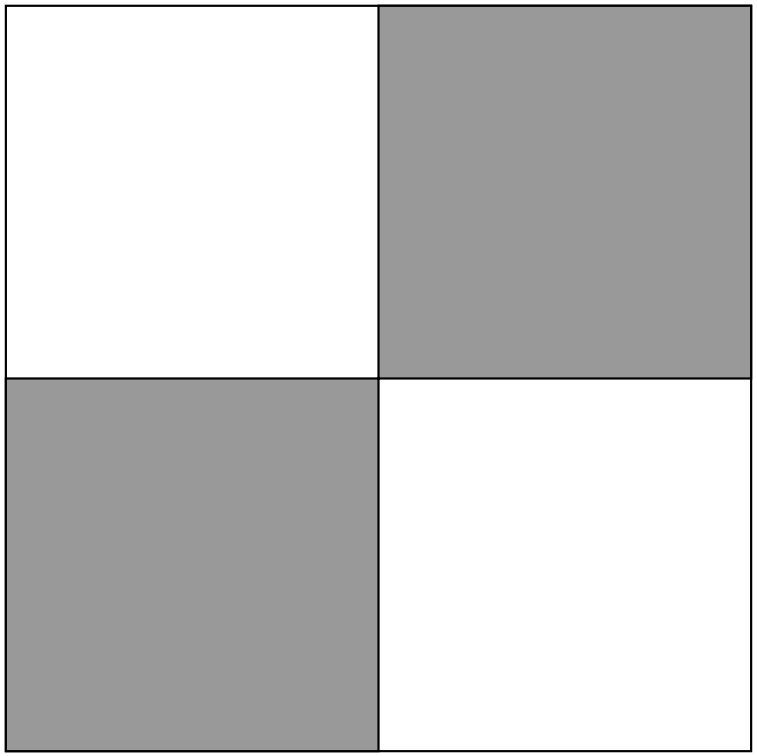
Critical density: $8\sqrt{2(\sqrt{2}-1)}+6\sqrt{2}-15 \approx 76.67\%$

Approximation factor: ≈ 1.3043

Rubies in a square: Examples



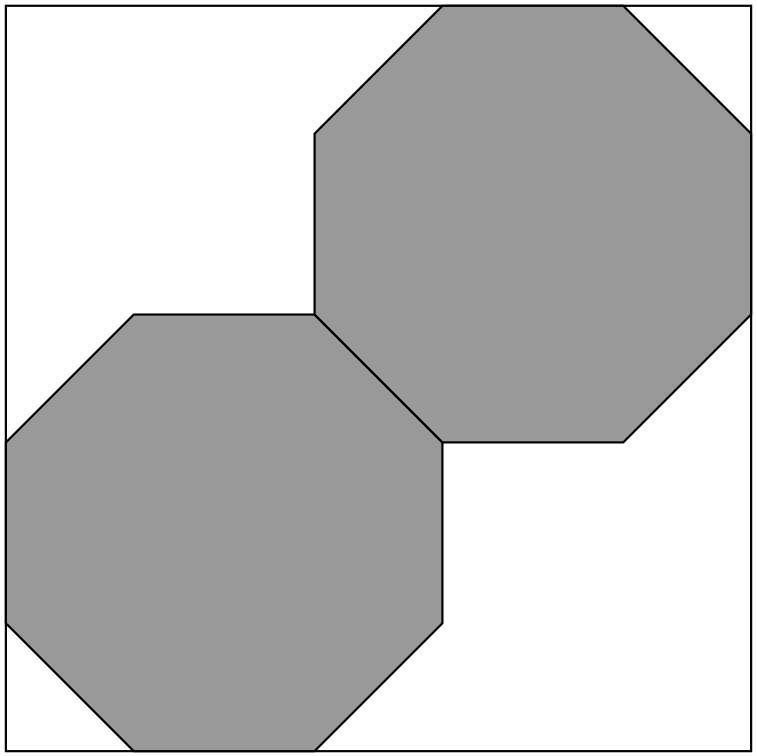
Squares in a square



Critical density: 50%

Approximation factor: 2

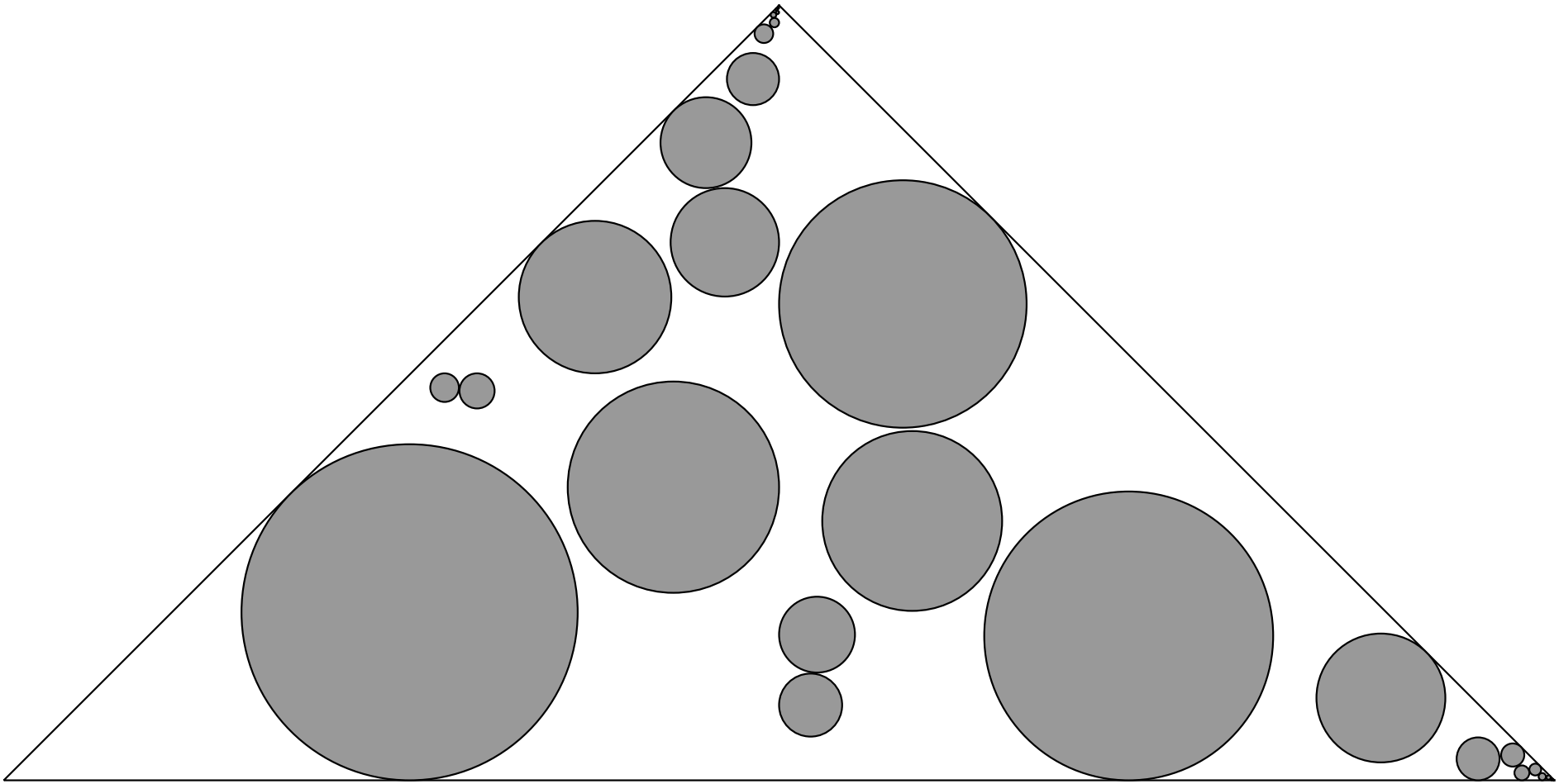
Octagons in a square



Critical density: $8(5\sqrt{2} - 7) \approx 56.85\%$

Approximation factor: ≈ 1.7589

Other shapes in an isosceles right triangle?



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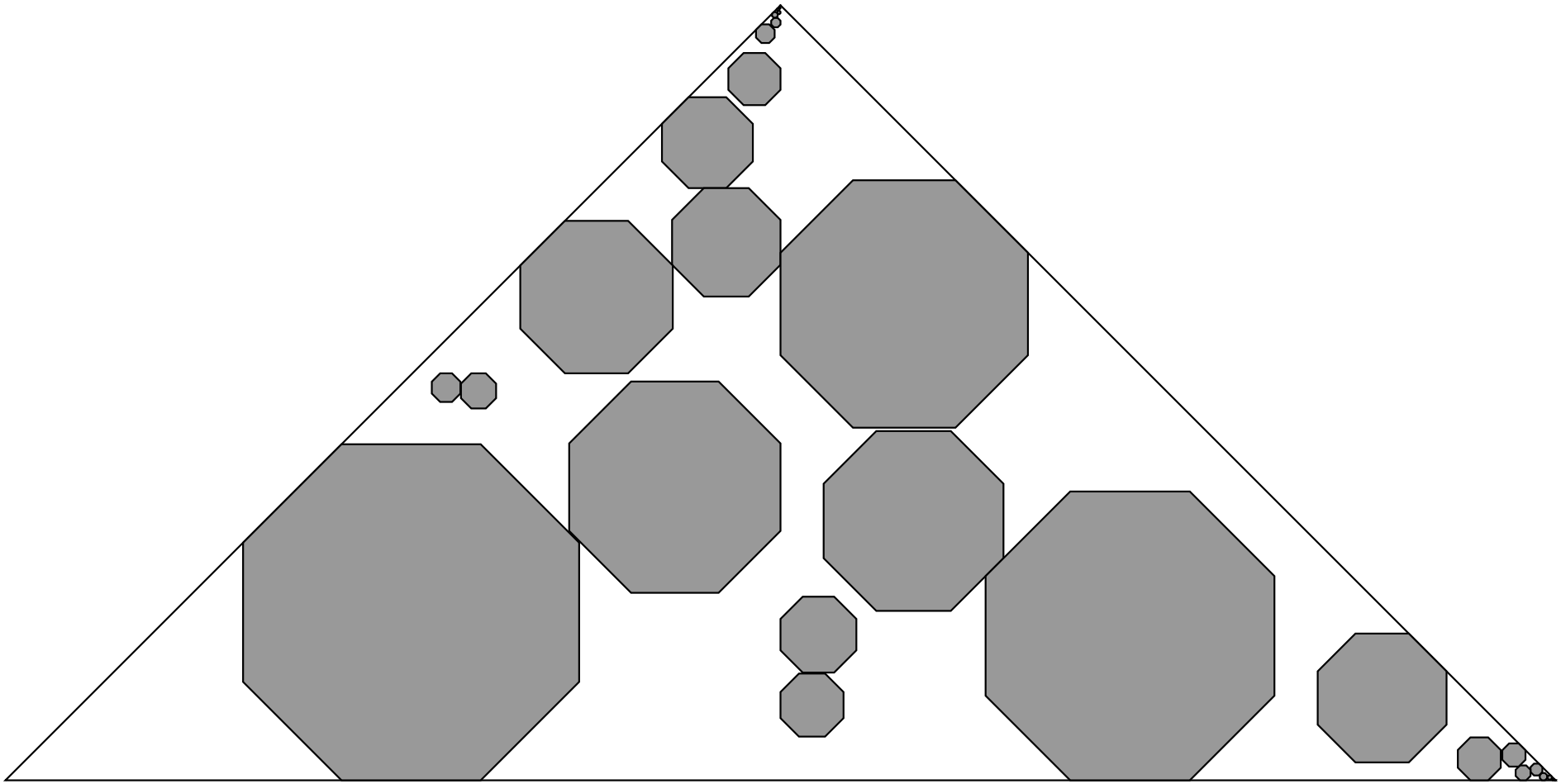

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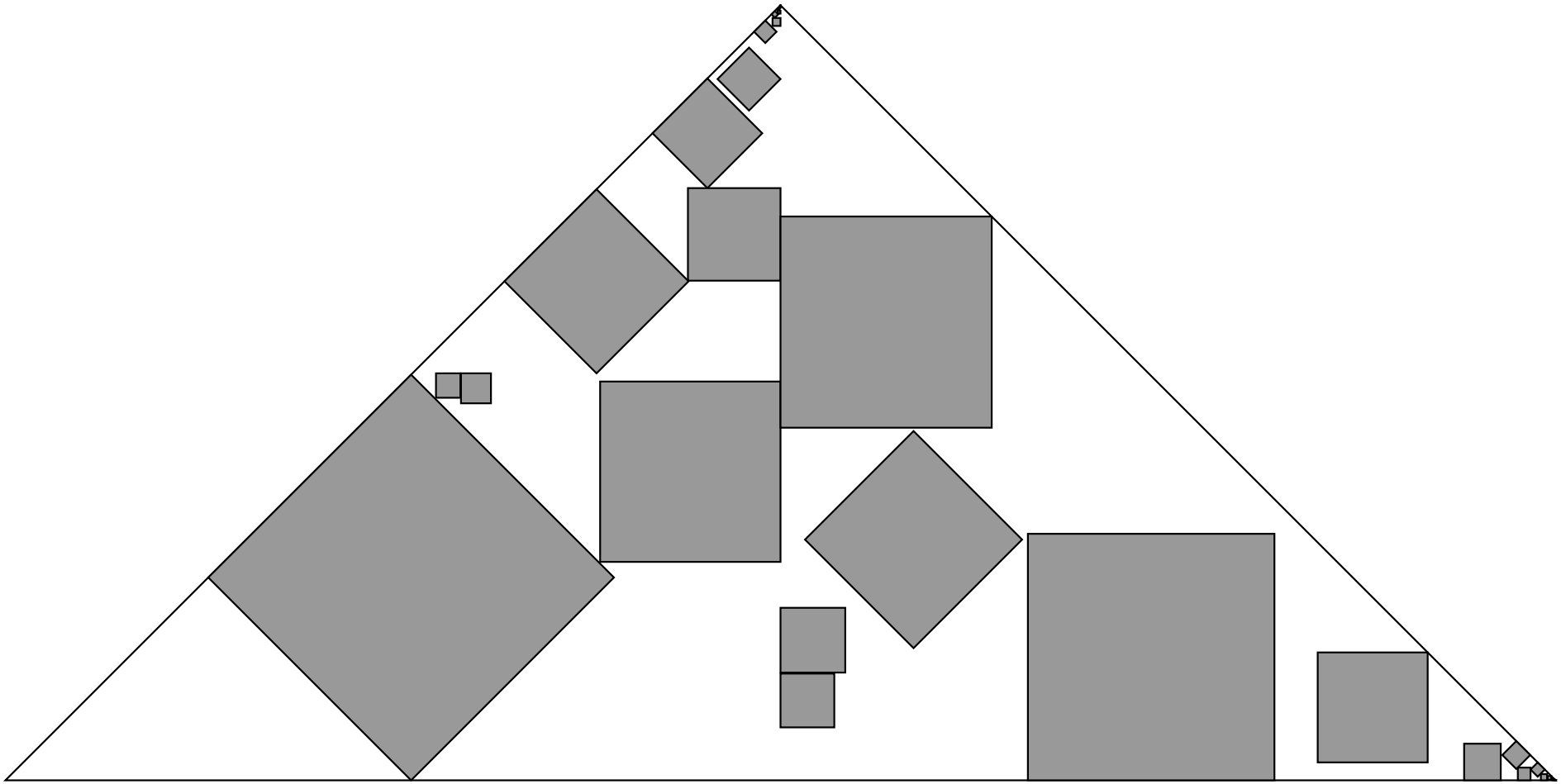

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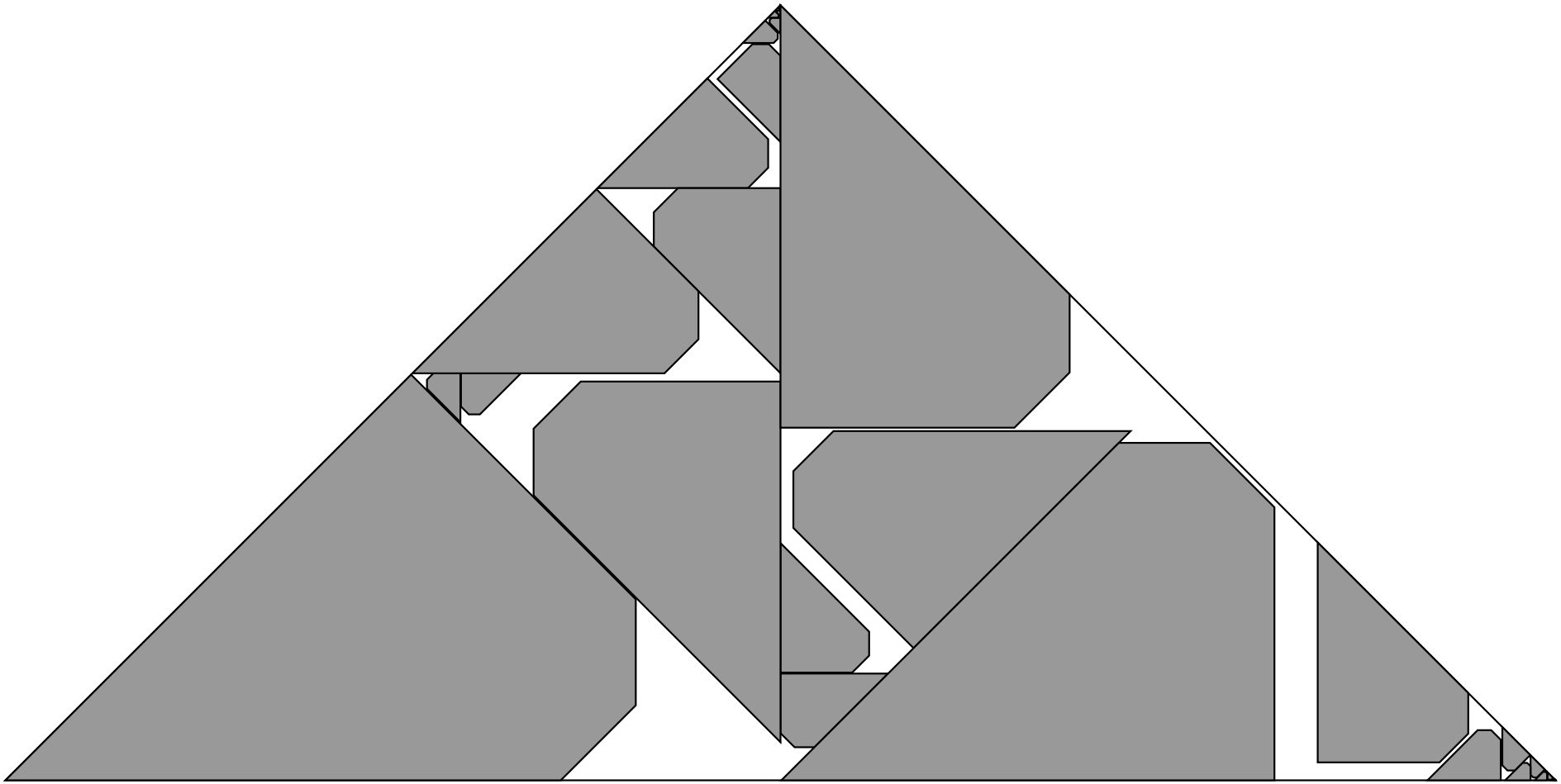
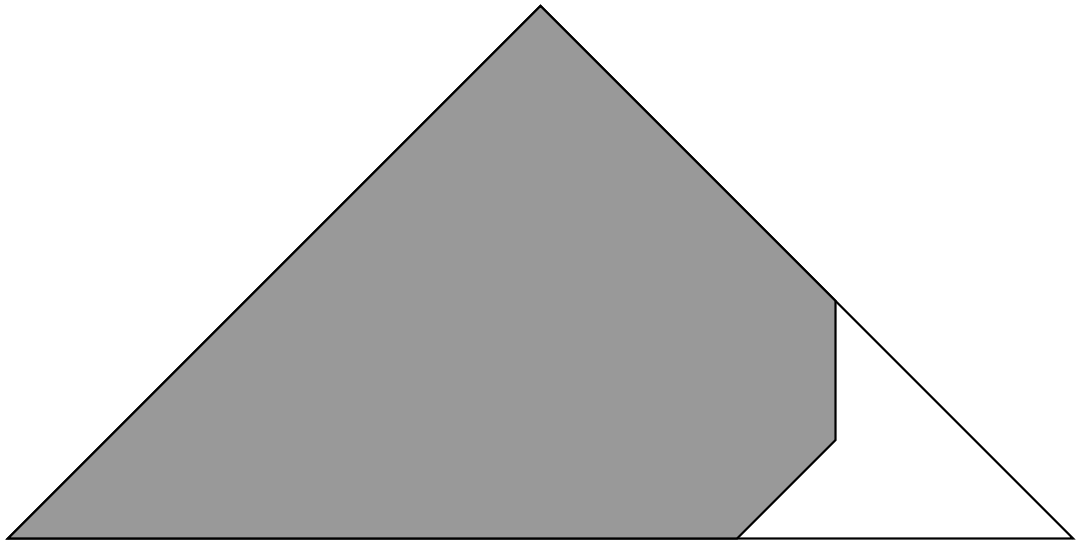


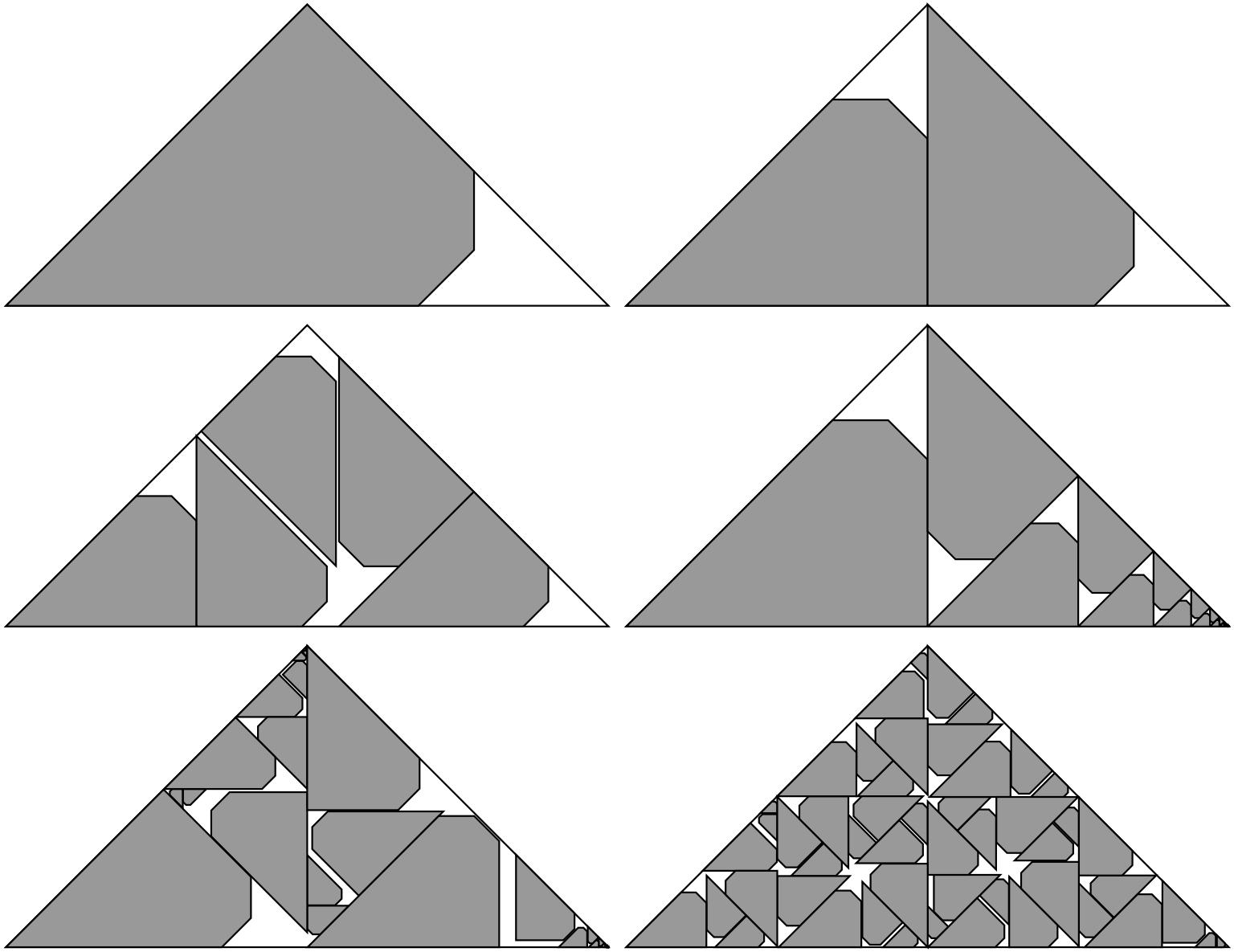
“Sharp rubies” in an isosceles right triangle



Critical density: $4\sqrt{2(\sqrt{2}-1)}+3\sqrt{2}-7 \approx 88.34\%$

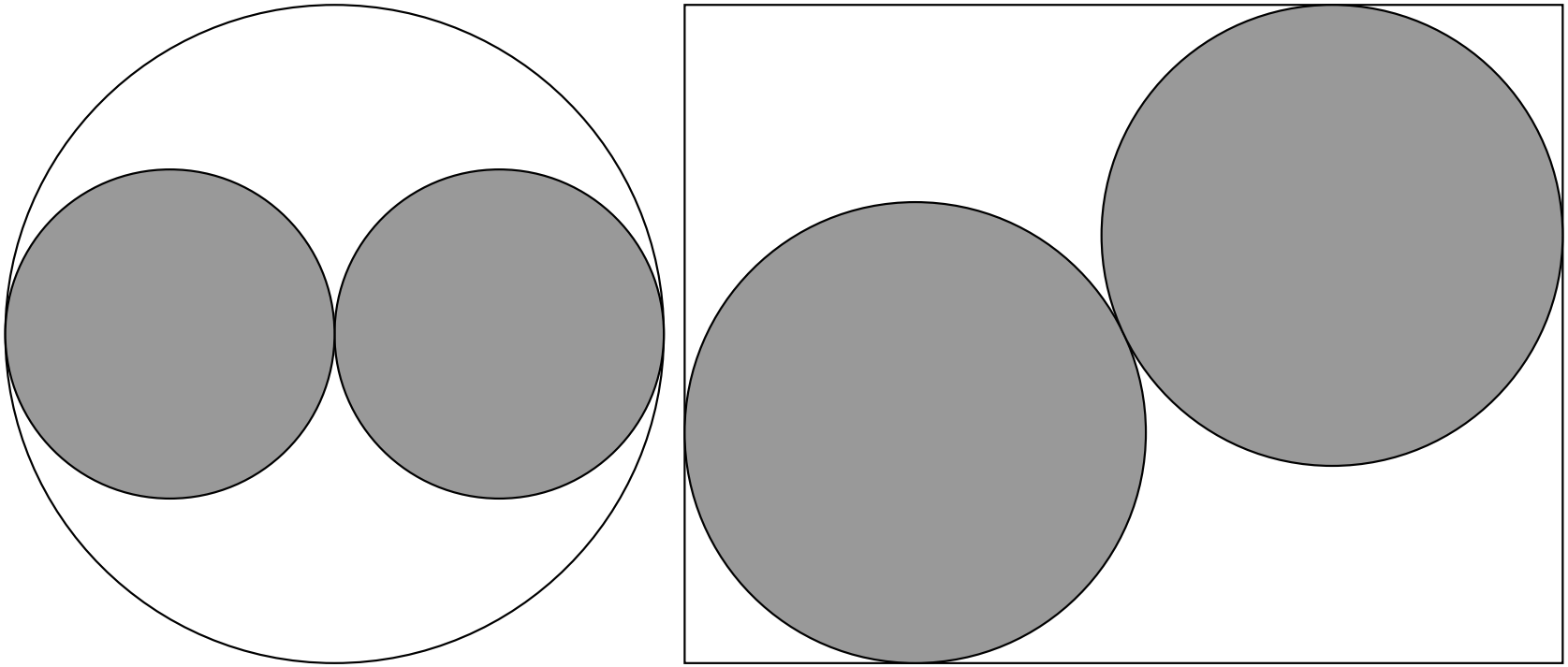
Approximation factor: ≈ 1.1320

“Sharp rubies” in an isosceles right triangle: Examples

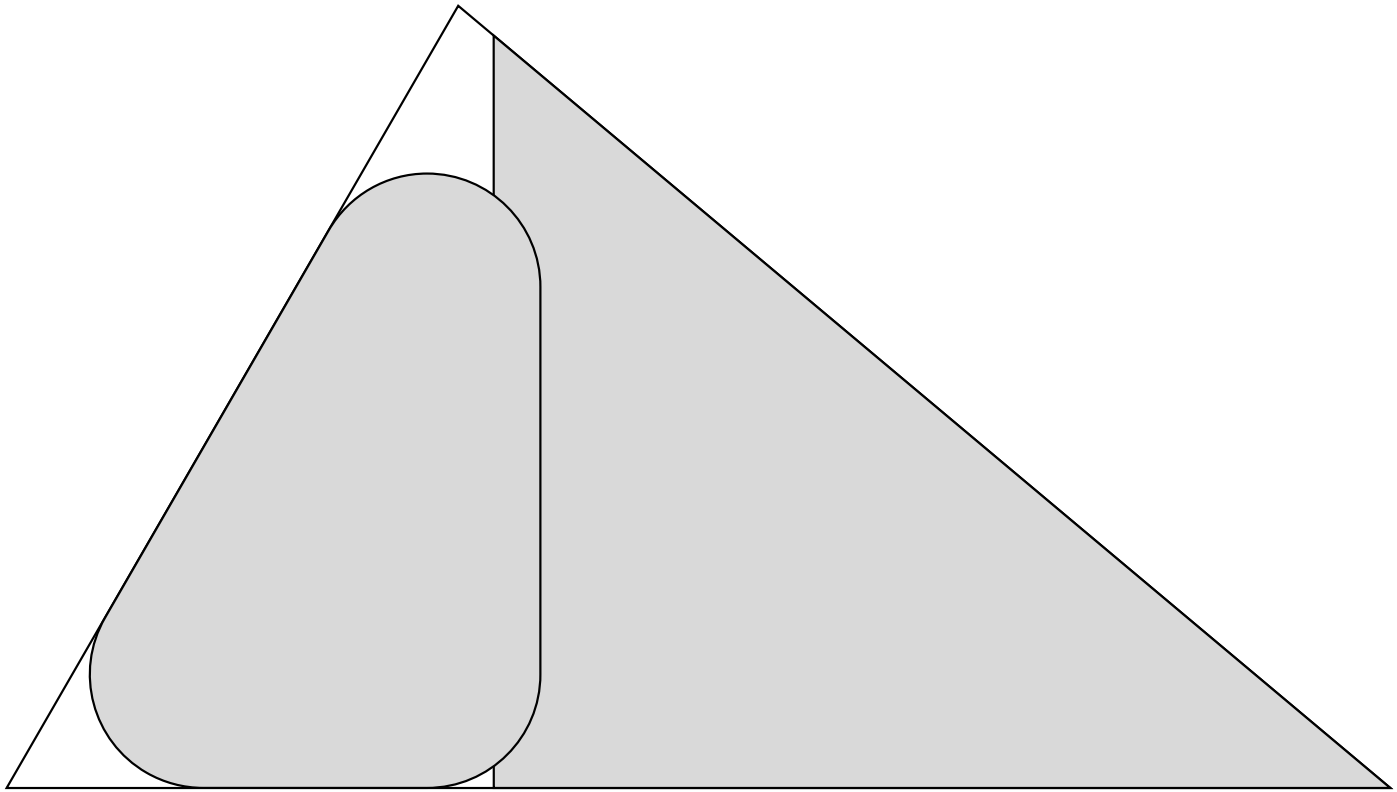


Outline

Future work: More container types



Future work: Acute triangles

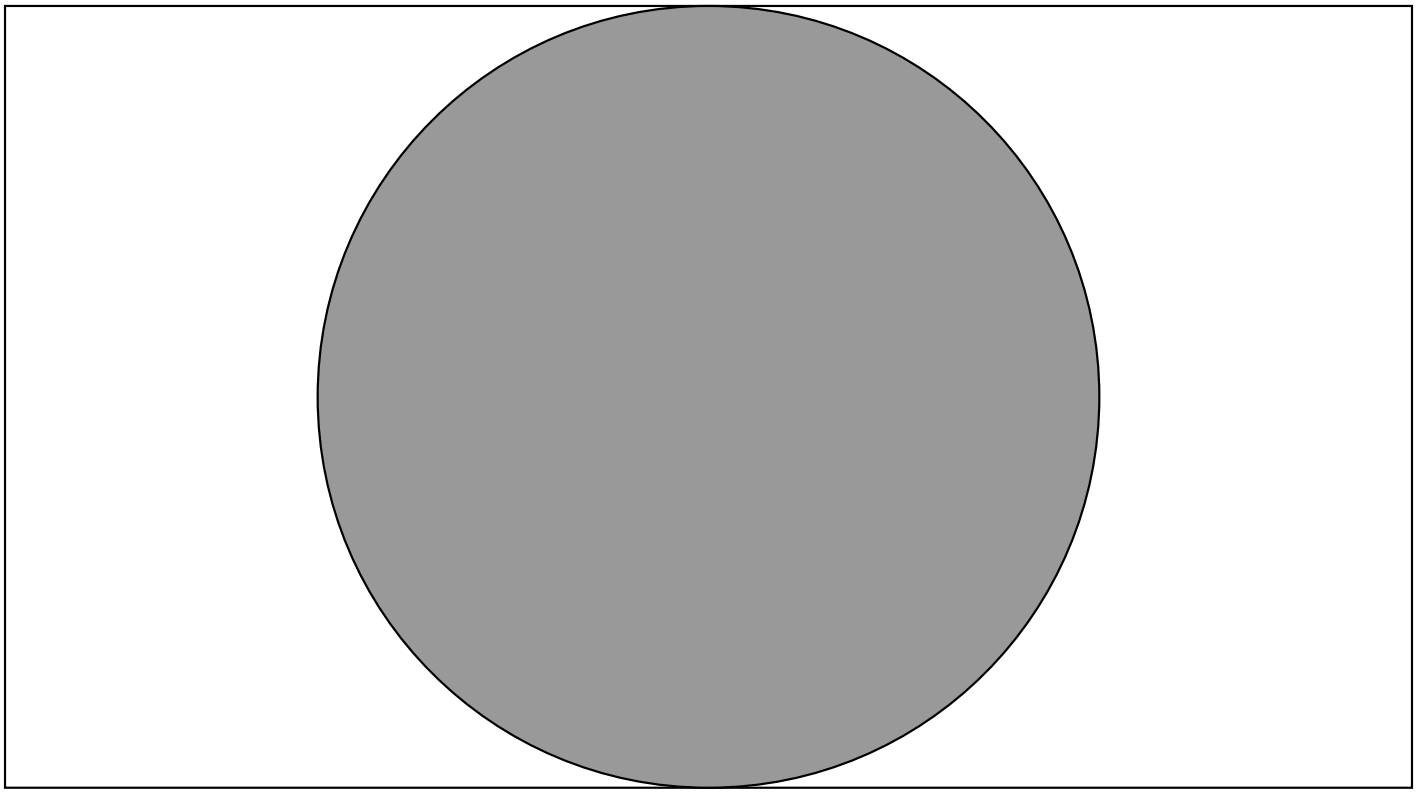


Future work: More object types

- Ovals
- Rectangles
- General convex polygons?

What do the critical instances look like?

Future work: Maximum object size

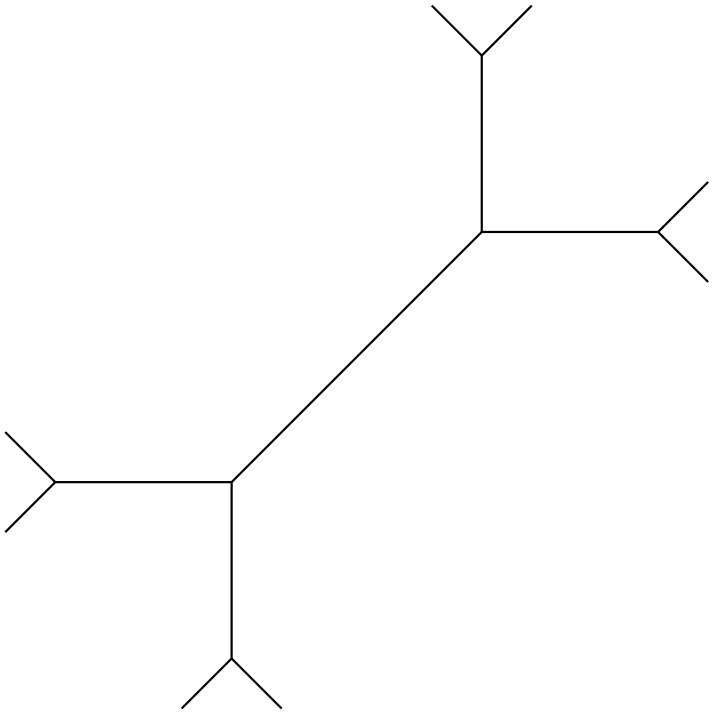


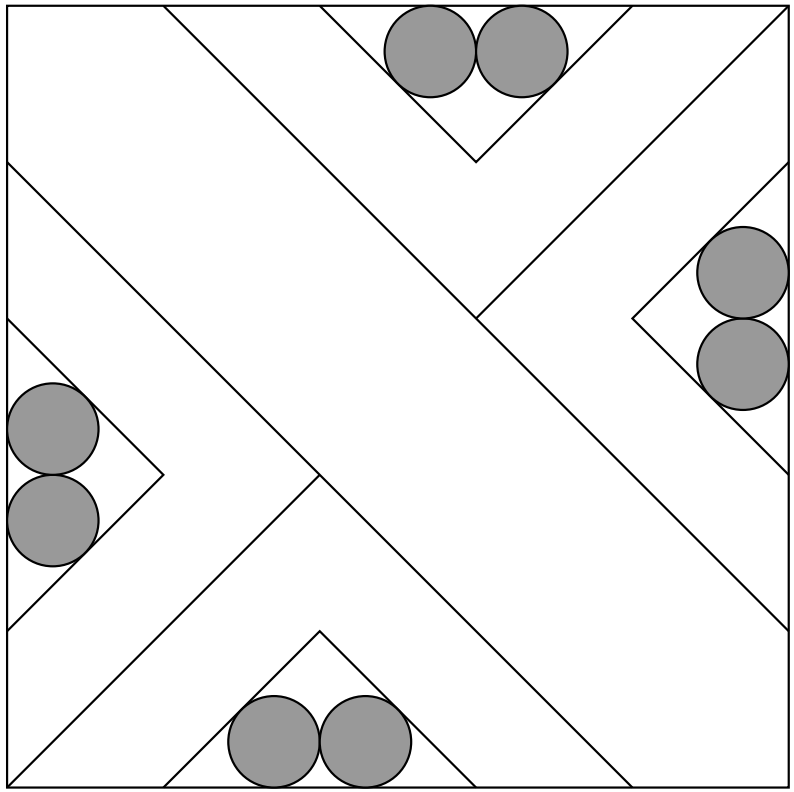
Future work: Online

Current best achievable density for packing squares into a square in an online setting: $2/5$

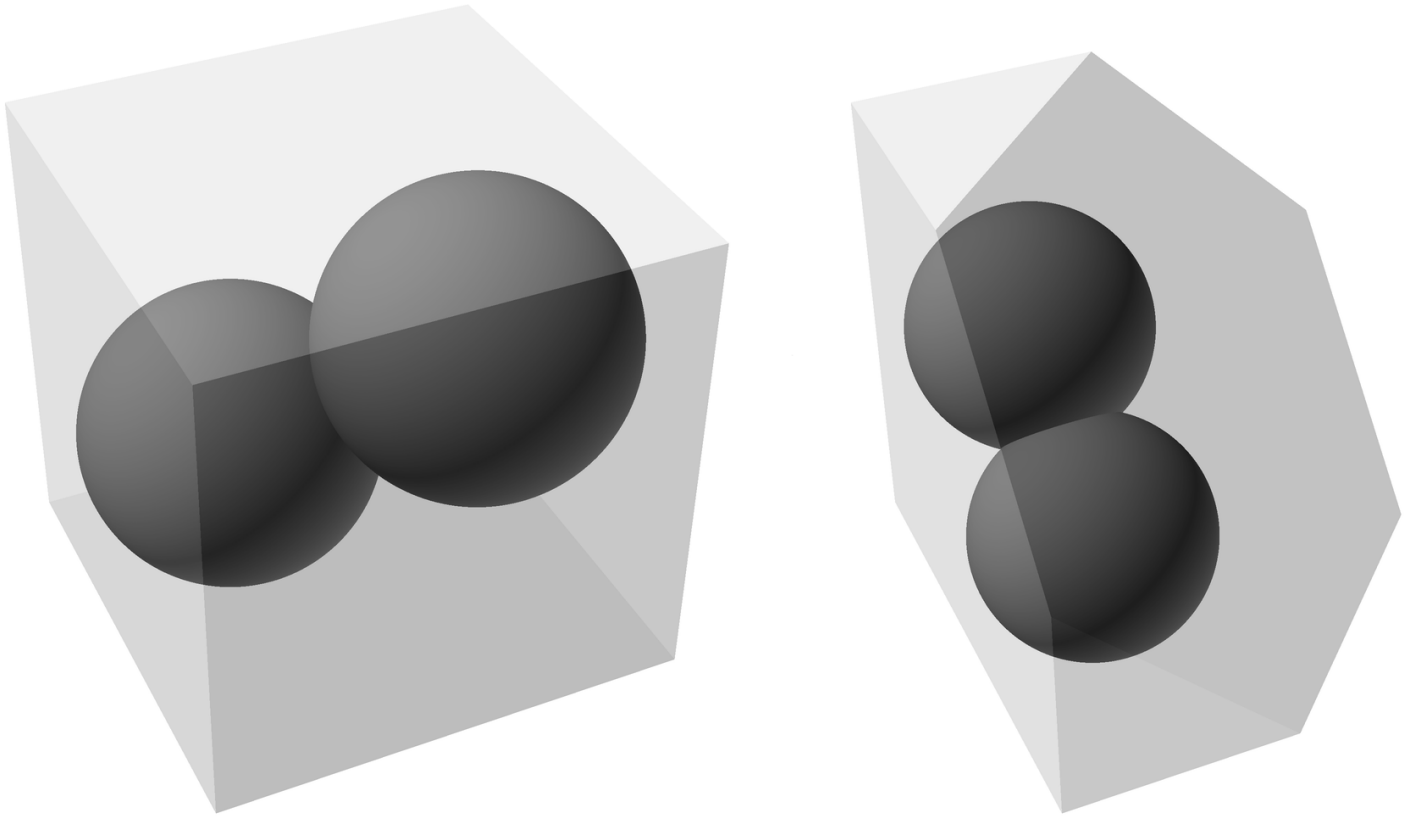
[Brubach 2015]

Future work: Circle/river packing

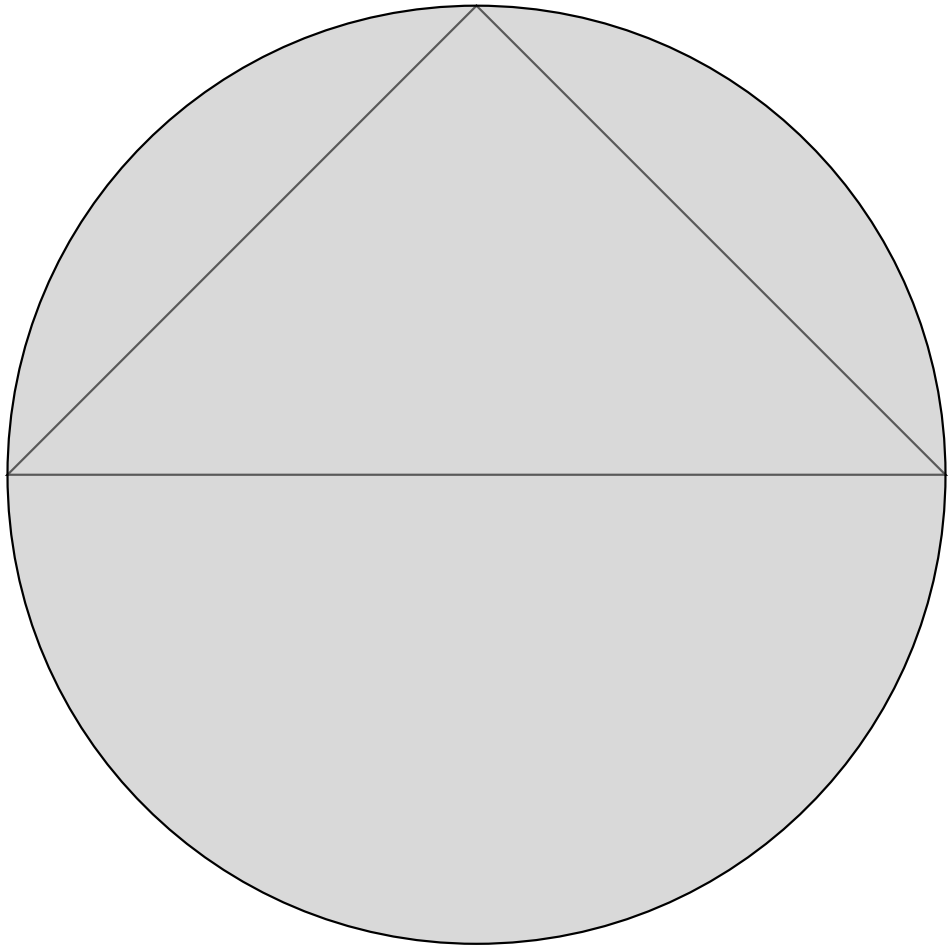




Future work: 3D



Future work: Covering



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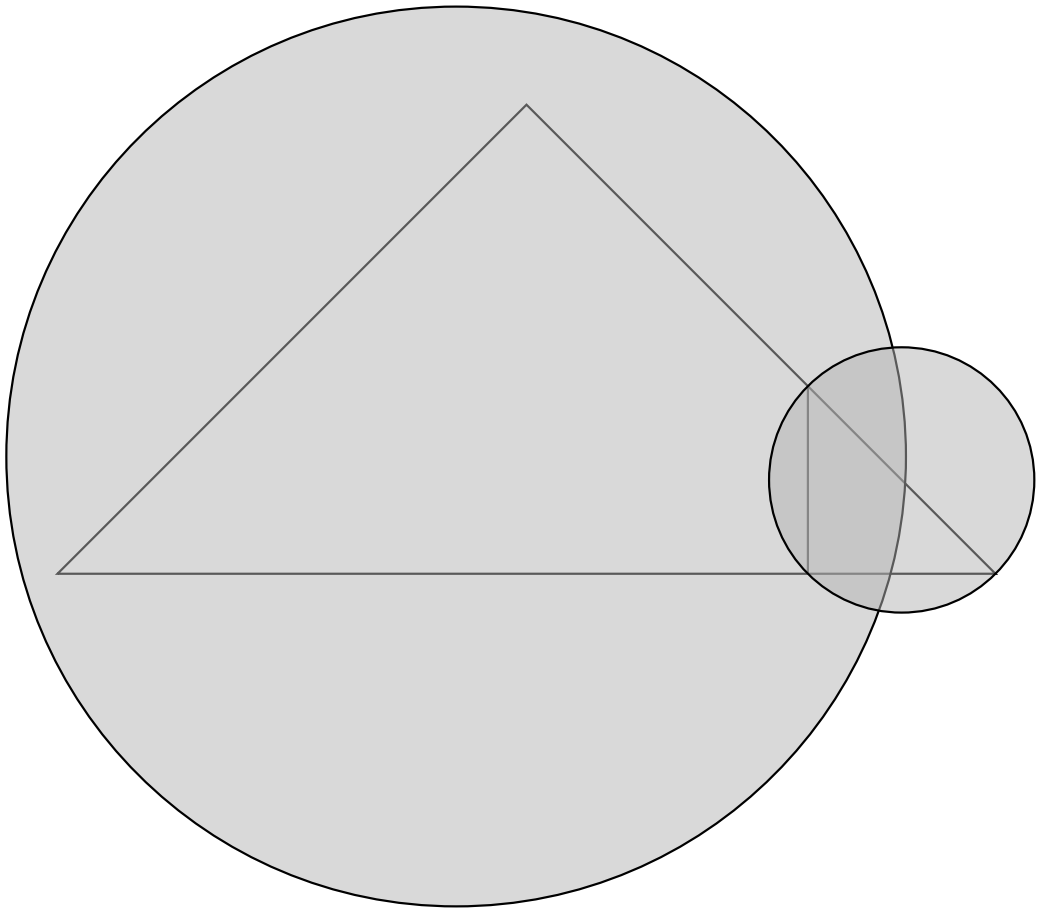

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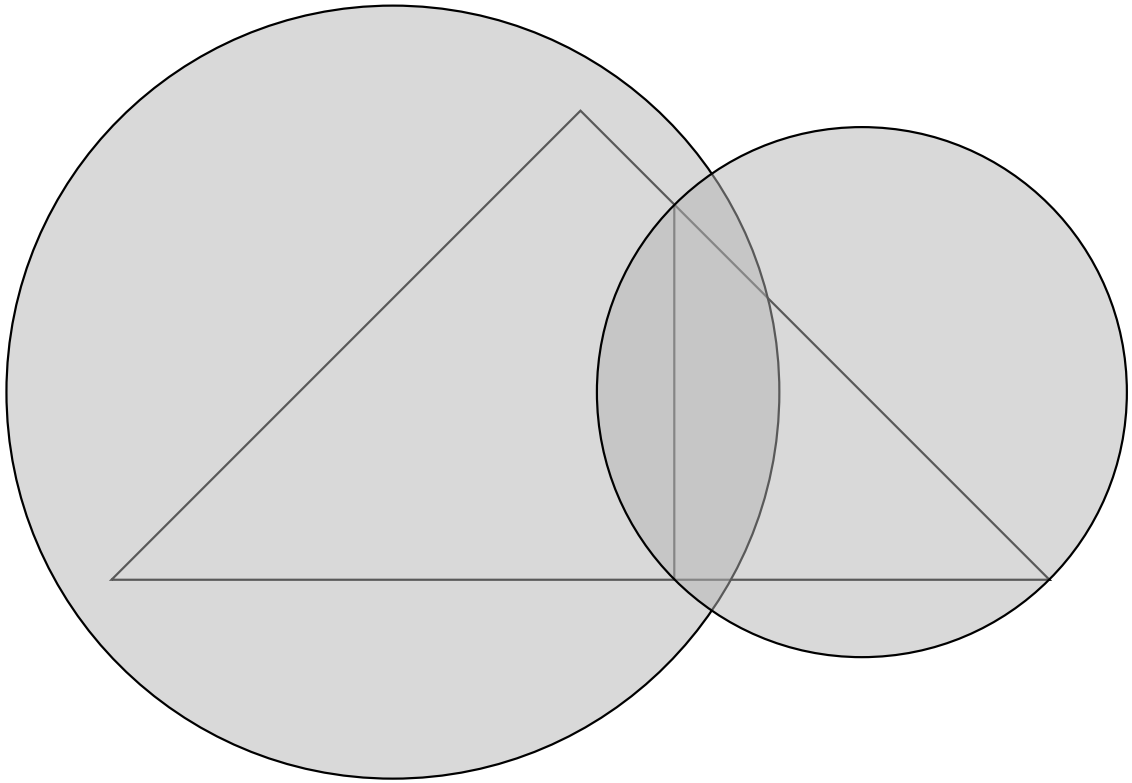

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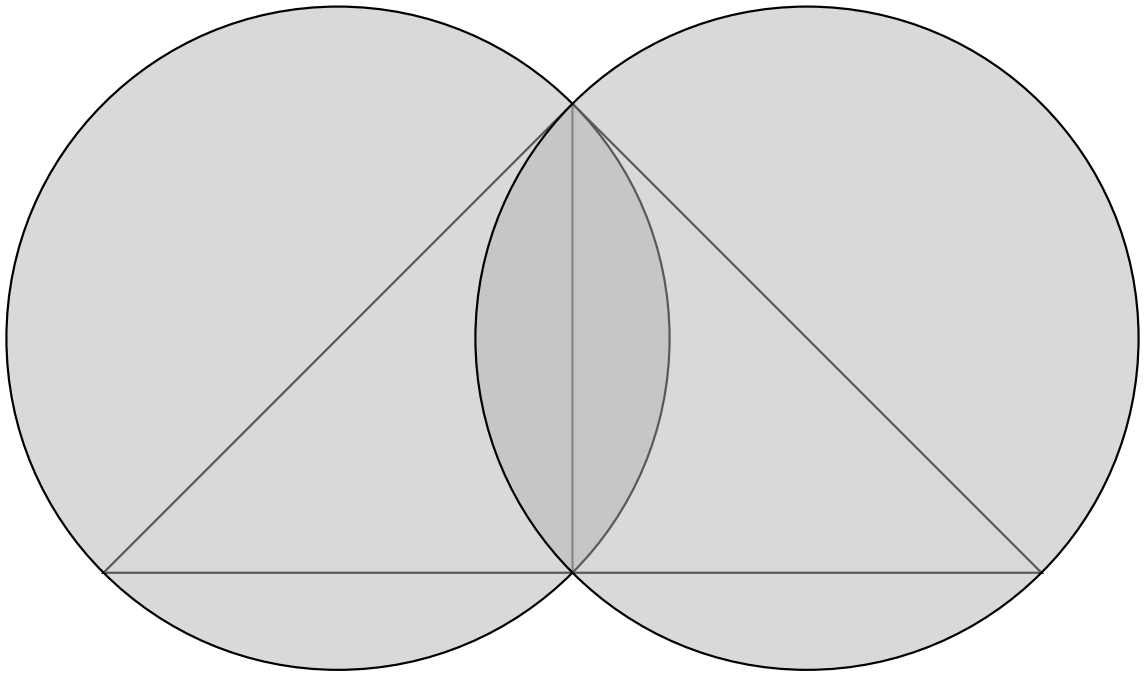

Contributions

1. Algorithms for packing. . .

Contributions

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   - circles, squares, and octagons into squares

Contributions

1. Algorithms for packing...
   - circles, squares, and octagons into squares
   - circles into non-acute triangles, thick isosceles triangles, and long rectangles

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   ...with *critical density*!

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2. Constant-factor approximation algorithms for these problems

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Thanks!

Bonus slide: Applications